A COMPENDIUM OF SPHERICAL ASTRONOMY

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OF

SPHERICAL ASTRONOMY

WITH ITS APPLICATIONS TO THE DETERMINATION AND REDUCTION OF POSITIONS OF THE FIXED STARS

BY

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PREFACE.

THE present volume is the first of a projected series having the double purpose of developing the elements of Practical and Theoretical Astronomy for the special student of the subject, and of serving as a handbook of convenient reference for the use of the working astronomer in applying methods and formulae. The plan of the series has been suggested by the author's experience as a teacher at the Johns Hopkins University, and as an investigator. The first has led him to the view that the wants of the student are best subserved by a quite elementary and condensed treatment of the subject, without any attempt to go far into details not admitting of immediate practical application. As an investigator he has frequently been impressed with the amount of time consumed in searching for the formulae and data, even of an elementary kind, which should be, in each case, best adapted to the work in hand.

The most urgent want which the work is intended to supply is that of improved methods of deriving and reducing the positions and proper motions of the fixed stars. Modifications of the older methods are made necessary by the long period, 150 years, through which positions of the stars now have to be reduced, and by the extension of astrometrical and statistical researches to a great and constantly increasing number of telescopic stars. Especial attention has therefore been given to devising the most expeditious and rigorous methods of trigonometric reduction of star positions, and to the construction of tables to facilitate the work.

Other features of the work are: A condensed treatment of the theory of errors of observation and of the method of least squares; an attempt to present the theory of astronomical refraction in a concise and elementary form without detracting from rigour of treatment; a new development of the theory of precession, now rendered necessary by the long period through which star places have to be reduced; the basing of formulae relating to celestial coordinates on the new values of the constants now used in the national ephemerides; a concise development of the rigorous theory of proper motions; the trigonometric reduction of polar stars to apparent place, and the development of what the author deems the most advantageous methods of correcting and combining observed positions of stars as found in catalogues.

Although the theory of astronomical instruments is not included within the scope of the present work, it is necessary, in using star catalogues, to understand the methods of deriving the results therein found from observations. The principles of the ideal transit instrument and meridian circle, omitting all details arising from imperfections of the instrument, are elegant and simple, and at the same time sufficient for the purpose in question. They are therefore briefly set forth in the chapter on deriving mean positions of stars from meridian observations.

A pedagogical feature of the work is the effort to give objective reality to geometric conceptions in every branch of the subject. The deduction of results by purely algebraic processes is therefore always supplemented, when convenient, by geometric construction. Whenever such a construction is represented on the celestial sphere, the latter is, in the absence of any reason to the contrary, shown as seen from the centre, so that the figure shows the sky as one actually looks up at it. Exceptions to this are some times necessary when planes and axes of reference have to be studied in connection with their relation to the sphere.

A similar feature, which may appear subject to criticism, is the subordination of logical order of presentation to the practical requirements of the student mind. While the method of first developing a subject in its general form and then branching out into particulars has been followed whenever it seemed best so to do, there are many cases in which special forms of a theory are treated in advance of the general form, the object being to prepare the mind of the student for the more ready apprehension of the general theory.

On the other hand, in order to lessen discontinuity of treatment, the policy has been adopted of relegating to an Appendix all the tables and many of the formulae of which most use is made. The choice of subjects for the Appendix is made from a purely practical point of view, the purpose being to include those tables, formulae, and data of most frequent application.

The "Notes and References" at the end of most of the chapters do not aim at logical or practical completeness. They embody such matters of interest, historical or otherwise, and such citations of literature, as the author hopes may be most useful to the student or the working astronomer. The list of Star Catalogues of precision at the end of the last chapter is, however, intended to be as complete as it was found practicable to make it; but even here it may well be that important catalogues have been overlooked.

The habit on the part of computers of using logarithmic tables to more decimals than are necessary is so common that tables to three decimals only are not always at hand. The Appendix therefore concludes with three-place tables of logarithms and trigonometric functions. These will suffice for the ordinary reduction of stars to apparent place, and many similar computations which have to be executed on a large scale.

WASHINGTON, *March*, 1906.

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$\equiv$, the symbol of identity, signifying that the symbol following it is defined by words or expression preceding it. It may commonly be read "*which let us call.*"

D_t, a derivative as to the time, expressing the rate of increase of the quantity following it.

$\odot$, sun's true longitude.

In the following list of symbols only those significations are given, which are extensively used in the work. Those used only for a temporary or special purpose are omitted.

Roman-Italic alphabet.

a, semi-major axis of an ecliptic orbit; the equatorial radius of the earth; also, reduced R.A., defined on p. 266.

b, polar radius of the earth; barometric pressure; latitude of a heavenly body.

c, earth's compression.

a, b, c, d are used to denote the Besselian star-constants. Chap. XI.

e, probable error; eccentricity.

f, ratio of apparent to geocentric distance.

g, intensity of gravity.

h, seconds of time in unit radius; also, west hour-angle.

k, angle between two positions of the plane of the ecliptic, or of the pole of the ecliptic.

l, the rate of general precession, annual or centennial.

m, the factor of $\tan z$ in the expression for the refraction; also, the constant part of the reduction of the R.A. of a star for precession.

m, the annual rate of precession in Right Ascension; m_c, the centennial rate $= 100m$.

n, the annual rate of motion of the celestial pole: $n_c = 100n$, the centennial motion.

N_0, supplement of the longitude of the instantaneous axis of rotation of the moving ecliptic; also, the angle which the direction of proper motion makes with the hour-circle of a star.

N_1, supplement of the longitude of the node of the ecliptic.

p, speed of luni-solar precession on the fixed ecliptic of the date; also, a quantity used in star-reductions (p. 267).

P, the absolute constant of precession.

r, radius vector.

s, angular semidiameter of a planet; also, angular distance.

t, time expressed in years or shorter units; also, mean time.

T, time expressed in terms of a century as the unit.

v, linear velocity, especially of a star, or of the earth in its orbit; also, angle of the vertical.

V, velocity of light.

w, weight of an observation or result.

z, zenith distance.

Greek alphabet.

α, Right Ascension.

β, latitude, referred to the ecliptic.

α, β, γ, angles made by a line with rectangular axes.

δ, Declination; symbol for increment or correction.

Δ, symbol of increment, of error, or of correction; distance of planet from the earth.

ϵ, obliquity of the ecliptic; mean error.

ζ, ζ_0, angles defining the relative positions of the mean equator and equinox at two epochs. §§ 127–130.

θ, angle between two positions of the mean equator.

κ, constant of aberration; also, speed of angular motion of the ecliptic.

λ, ecliptic longitude; terrestrial longitude; also, planetary precession.

μ, proper motion of a star; also, index of refraction of air.

μ_α, " " in Right Ascension.

μ_δ, " " in Declination.

π, parallax; ratio of circumference to diameter.

ρ, distance from centre of earth; radius of earth.

τ, temperature above absolute zero; also, sidereal time.

ψ, total luni-solar precession on an initial fixed ecliptic.

ϕ, astronomical latitude of a point on the earth's surface.

ϕ', geocentric latitude " " "

Ω, longitude of the moon's node.

PART I.

PRELIMINARY SUBJECTS.

CHAPTER I.

INTRODUCTORY.

THIS opening chapter is devoted to certain preliminary matters which can better find a place here than elsewhere. The beginner in astronomical work may be accustomed only to those modes of mathematical thought and investigation which are formally rigorous. He has now to enter a field in which, owing to the concrete form of the subject-matter, he must frequently be satisfied with approximations to a rigorous result, and a consequent abatement of the strictness of mathematical demonstration. One must learn to work in this field without sacrificing rigor of thought, or losing sight of the possible deviations of the results from the ideal truth. To do this, we give examples of the most common cases of deviation from formal precision in astronomical practice.

1. Use of finite quantities as infinitesimals.

The omission of all powers of a small quantity above the first is very common in the mathematical methods of astronomy. In this case we are said to treat the quantity as an infinitesimal. The practice rests on the following basis:

Let u be a function of x,

$$u = \phi(x),$$

and let us assign to x an increment Δx, and call Δu the corresponding increment of u. The new value of u will be

$$u + \Delta u = \phi(x + \Delta x). \qquad (1)$$

Developing by Taylor's theorem, we have

$$\Delta u = \frac{du}{dx}\Delta x + \frac{1}{2}\frac{d^2u}{dx^2}\Delta x^2 + \ldots.$$

If Δx is below a certain limit of magnitude, and the differential coefficients $\frac{du}{dx}$ etc. not too great, the second and following terms of this development may be omitted. For example, let Δx be $50'' = 0{\cdot}000\,24$ in arc. Reduced to seconds the square will be

$$\Delta x^2 = 0''{\cdot}012.$$

In much astronomical work an error of $0''{\cdot}01$ is quite unimportant; indeed cases are frequent in which we need not consider a correction so small as $0''{\cdot}1$ or even an entire second. We may extend and generalize this conclusion as follows:

If the second derivative does not exceed unity, we may use the equation

$$\Delta u = \frac{du}{dx}\Delta x, \quad\quad (2)$$

dropping the higher terms of the series

whenever $\Delta x < 50''$ *if an error of* $\pm 0''{\cdot}01$ *is unimportant,*
" " $< 150''$ " " $\pm 0''{\cdot}1$ " "
" " $< 500''$ " " $\pm 1''{\cdot}0$ " "

If the second derivative exceeds unity, the limits must be reduced in a like proportion.

2. Use of small angles for their sines or tangents.

The general rule embodied in (2) leads to the constant use of small angles themselves instead of their sines or tangents, and to putting their cosines equal to 1. We have, by well-known developments,

$$\sin s = s - \tfrac{1}{6}s^3 + \ldots,$$
$$\tan s = s + \tfrac{1}{3}s^3 + \ldots.$$

Hence, to terms of the third order in s, we may use the forms

$$\left.\begin{aligned}\sin s &= s(1 - \tfrac{1}{6}s^2)\\ \tan s &= s(1 + \tfrac{1}{3}s^2)\end{aligned}\right\}, \quad\quad (3)$$

$$\left.\begin{aligned}s &= \sin s(1 + \tfrac{1}{6}\sin^2 s)\\ s &= \tan s(1 - \tfrac{1}{3}\tan^2 s)\end{aligned}\right\}. \quad\quad (4)$$

These equations presuppose that the angle is expressed in circular measure, the radian being the unit. But, in actual computation, the unit is nearly always the degree, minute, or second, the last being the usual unit for small angles. The fact that s is expressed in seconds may be indicated by two accents. When so expressed, it may be reduced to circular measure by multiplication by the angle of 1″ expressed in that measure, which is practically the same as sin 1″. Thus we have

$$s = s'' \sin 1''.$$

There being 206 265″ (more exactly 206 264″·806) in the radian, we have

$$s'' = 206\,265''\, s = [5{\cdot}314\,425]\, s,$$

the number in brackets being the logarithm of the factor. This form of expressing multiplication by a factor whose logarithm only is given is very common. So, putting R for the number 206 264·806, we may write instead of (4)

$$\left.\begin{aligned} s'' &= R'' \sin s(1 + \tfrac{1}{6}\sin^2 s) \\ s'' &= R'' \tan s(1 - \tfrac{1}{3}\tan^2 s) \end{aligned}\right\} \quad \text{(5)}$$

If we have a series containing various powers of a small angle, the practically easiest method of manipulating it is to reduce one factor of each term to seconds, and retain the others in the general form. For example, the general form

$$s = a + bs^2 + cs^3 \ldots$$

becomes, in seconds, $\quad s'' = a'' + bs''s + cs''s^2 \ldots,$

where $s'' = R''s$, $a'' = R''a$, while s and s^2 are expressed in radians.

Below a certain limit we may drop the factor $\sin^2 s$ from the equations (3) and (4). This limit is that below which the product

$$\tfrac{1}{6}s'' \times \sin^2 s \quad \text{or} \quad \tfrac{1}{3}s'' \times \tan^2 s$$

is too small to affect our result. If an error of 0″·01 is unimportant, the upper limit for s'' falls below that value for which

$$\tfrac{1}{3}s'' \times \sin^2 s = 0''{\cdot}01,$$

which gives $\quad \sin^2 s = \dfrac{0''{\cdot}03}{s''}.$

We may also use instead of (4) or (5),

$$\sin s = \frac{s''}{R}.$$

Equating the square of this equation to the preceding one gives

$$s''^3 = 0{\cdot}03R^2,$$

which gives for s a value somewhat exceeding 1000″. The general rule therefore is that, in using any angle below this limit, the sine, tangent, and angle may be used indifferently.

The putting of 1 for the cosine of a small angle is governed by similar considerations. The cosine of 1000″ differs from 1 by less than 1 : 85 000. Hence if, in an expression of the form

$$A \times \cos s,$$

an error $0{\cdot}000\,012A$ is of no importance, we may always suppose

$$\cos s = 1 \quad \text{if} \quad s'' < 1000''.*$$

3. Unavoidable errors in computation.

We cannot determine a physical quantity with mathematical exactness. Measures of length, weight, volume, and every other magnitude are liable to errors, which we may reduce more and more by laborious attention to details, but can never absolutely eliminate. Many measures may be in error by their thousandth part, and it is only a few fundamental quantities which we can consider as known within their millionth part. Even were a rigorous determination possible, its rigorous expression by any system of numbers would not be. Our systems of expressing quantities numerically by an infinite series, proceeding according to the diminishing powers of a base, is the best that can be applied in practice. In our traditional system of numeration the base is the number 10. Could we begin anew, the number 12 might be better; but this is impracticable. In a decimal expression we reduce the maximum error by one-tenth by every figure we add, but can only approach to the rigorous value of a concrete physical quantity, or the logarithm of a number which is not an integral power or root of 10. As a general

* Mention may here be made of the almost universal practice of using the word "arc" to indicate an angle expressed in degrees, minutes, or seconds.

rule, there is no use in adding decimals beyond the practically attainable limit of accuracy.

In a numerical computation, especially with logarithms, one should always have some idea of the degree of accuracy attainable or desirable; or, to speak with more precision, of the magnitude of the errors to which the data and results are liable. The accuracy of a result is limited by that of the data on which it depends, so that, in all computations, the result must be affected by errors arising from those of the data, no matter with what precision the computation is made. As every additional figure used in computation adds to the labour, the first question to be considered by the computer in entering upon a work is: How many figures are necessary in the logarithms in order that the unavoidable errors of the result may not be increased by the errors of the logarithms? The logarithmic tables in ordinary use range from three to seven decimals, and the question of the error arising from the decimals following the last being omitted is the first to be considered.

Let q be the true value of a quantity and $\pm\delta$ the error of the value we derive, so that $q\pm\delta$ is the value we reach by computation. We have to find what value of δ will arise from using logarithms from which the decimals after a certain order are dropped. Treating δ as an infinitesimal, we have, for the error of the logarithm, corresponding to the error δ of q,

$$\log(q+\delta)-\log q=\log\left(1+\frac{\delta}{q}\right)=M\frac{\delta}{q},\qquad(6)$$

M being the modulus, 0·434 29....

If we use n-place logarithms, the value of the unit in the last figure will be 10^{-n}. In taking out a logarithm, the error need be only a fraction of this unit; but in adding up several, it may reach or exceed the unit. Assuming a unit error in the last figure of $\log q$ we shall have

$$M\frac{\delta}{q}=10^{-n}$$

and

$$\delta=\frac{10^{-n}}{M}q.$$

Assigning to n the successive values 3 to 7 we have the corresponding errors of q as follows:

$$\left.\begin{array}{lll} \text{3-place logarithms,} & \delta = \pm \cdot 0023q \\ \text{4-} \quad " \quad " & \quad " = \pm \cdot 000\,23q \\ \text{5-} \quad " \quad " & \quad " = \pm \cdot 000\,023q \\ \text{6-} \quad " \quad " & \quad " = \pm \cdot 000\,002\,3q \\ \text{7-} \quad " \quad " & \quad " = \pm \cdot 000\,000\,23q \end{array}\right\} \dots\dots\dots\dots (7)$$

Using round numbers, we may say that the use of 3-place logarithms will give a result correct to the 400th part of its amount, 4-place logarithms to the 4000th part, and so on. Conversely, if we wish a result correct only to the 100th part, 3-place logarithms will do; to the 1000th part, 4-place; to the $1:10^n$ part we should use $n+1$ decimals in the logarithms.

4. The preceding rules give only a relation between the errors of a logarithm and of the corresponding number. The relation between the error of the data and of the result can be expressed thus: Let the given data be u, v, w, etc.; the quantity to be computed p. We may then regard p as a function of u, v, w. If we put δu, δv, δw, for the errors of these quantities, the error in p will be

$$\delta p = \frac{dp}{du}\delta u + \frac{dp}{dv}\delta v + \frac{dp}{dw}\delta w + \dots \dots\dots\dots (8)$$

If the values of the derivatives which enter into this expression are large numbers, the error of the result will be greater than those of the data in like proportion.

A case of this kind occurs in determining a small angle by its cosine, or one near 90° by its sine. The error of the angle may then be many times greater than that of the function by which it is determined. It is, therefore, preferable to determine an angle by its tangent when this can be done.

In ordinary computation a common case of this kind is that in which the result comes out as a difference of two large and nearly equal quantities, or as the quotient of two such differences, or of two small quantities. In such a case more logarithms must be used in computing the large numbers, or the small numbers must be carried to a higher degree of precision, than would be prescribed by the rule.

5. Derivatives, speeds, and units.

As the theoretical study of the differential calculus does not suffice for its practical applications, we begin with some remarks on derivatives and the units in terms of which they are expressed.

The derivative of a quantity with respect to the time, at any moment, represents the velocity or *speed of increase* of the quantity at that moment. If the increase is constant the speed is found by dividing the increment, whatever it may be, by the time necessary for that increment to take place. If, however, the speed of increase is continually varying, it is at any moment the quotient of the infinitesimal increase of the quantity by the infinitesimal time required for that increase. Using the notation of the differential calculus, if we put Q for the quantity and S for the speed we have

$$S = \frac{dQ}{dt} \equiv D_t Q.$$

In the present work we use the more compact symbol D_t to express the derivative, or the speed of increase of the quantity following it. When this symbol is written before any quantity Q the combination D_tQ therefore signifies the rate of increase of Q at any moment.

The next question concerns the units, especially of time, in which the speed is to be expressed. The fact that the latter is determined by an increase during an infinitesimal time sets no limit upon the length of the unit of time that may be used. The units employed in astronomy range all the way from one second to 100 years. The unit of speed is defined as that speed which, if it remained constant during the unit of time, would produce unit increase in the quantity whose speed is designated. In ordinary language we express these units whenever necessary, speaking, for example, of 5 *feet* per *second* or 15 *degrees* per *hour* or 20 *seconds* per *century*.

The relation of these units to the infinitesimals by which a derivative is defined needs a moment's consideration. If we say that the speed of increase of the R.A. of a star is 300 s. per century, this means that the R.A. would increase by that amount in a century if the speed remained constant. To determine or express the derivative, that is the speed, we may take, instead of an infinitesimal time dt, any interval during which the speed may practically be treated as constant. In the motions which affect the stars the general rule is that the speed during a year varies so little that the interval adopted may be a year. This has led to the use of symbols in a double meaning, which may sometimes lead to confusion if the difference in the two meanings is not understood. For example the symbol m is used to express the constant part of the precession in R.A. of a star during the year. As its variation during any one year is too minute to be considered, the annual speed of the precession in R.A. is also indicated by the symbol m. But it should be understood that these two interpretations are different, though the symbol, and the number it represents, may be the same.

6. Differential relations between the parts of a spherical triangle. In a large class of astronomical problems the given quantities are three of the parts of a spherical triangle, and the problem is to find one or more of the three remaining parts. As auxiliary to this problem it may be asked what changes or errors in the required part will be produced by given small changes, or errors treated as infinitesimal, in the three given parts. This requires that we find the derivatives of the required parts as to the given parts, the latter being treated as independent variables.

There are three independent relations, and no more, between the six parts. From these relations, expressed as equations, we may eliminate any two of the parts, leaving one equation between four parts, from which any one part may be determined in terms of the other three. Let such an equation be expressed in the form

$$\phi(x, y, z, u) = 0; \qquad \text{(9)}$$

we shall then have by differentiation

$$\frac{d\phi}{dx}dx + \frac{d\phi}{dy}dy + \frac{d\phi}{dz}dz + \frac{d\phi}{du}du = 0. \qquad \text{(10)}$$

From this equation the value of any one differential in terms of the three others may be found. For example

$$\frac{dx}{dy} = -\frac{\frac{d\phi}{dy}}{\frac{d\phi}{dx}}.$$

Since there are 15 combinations of 4 parts out of 6, we may write as many equations of this form. But, only three of these will be independent of each other, and these three may be formed from one by permutation of parts. Let us take the fundamental equation

$$\phi = \cos a \cos b + \sin a \sin b \cos C - \cos c = 0;$$

we shall have

$$\frac{d\phi}{da} = -\sin a \cos b + \cos a \sin b \cos C$$

$$= -\sin c \cos B, \quad \text{.......... (11)}$$

$$\frac{d\phi}{db} = -\cos a \sin b + \sin a \cos b \cos C$$

$$= -\sin c \cos A, \quad \text{.......... (12)}$$

$$\frac{d\phi}{dc} = \sin c, \quad \text{.......... (13)}$$

$$\frac{d\phi}{dC} = -\sin a \sin b \sin C. \quad \text{.......... (14)}$$

The equation between the four differentials may, therefore, be written:

$$\sin c \cos B da + \sin c \cos A db - \sin c dc + \sin a \sin b \sin C dC = 0. \quad (15)$$

From this two others of like form may be written by changing each letter into the one next following in alphabetical order, A and a following C and c.

The forms derived from these for practical application will be found in Appendix I.

7. Differential spherical trigonometry.

In using the differential increments of the parts of a spherical triangle, and of angles and arcs generally on the sphere, a great advantage is often gained by treating the subject geometrically. The following are fundamental theorems at the base of the method:

(i) *An infinitesimal spherical triangle may be treated as a plane triangle when infinitesimals of an order higher than the second are neglected.*

This follows from the fact that the excess of the sum of the angles over 180°, being proportional to the area of the triangle, is of the second order; and that the deviations of the parts from those of a corresponding plane triangle formed by the projection of the spherical triangle on a tangent plane to the sphere at the place of the triangle are of the second order.

It must be noted, however, that this theorem presupposes the three angles to be finite quantities, and is, therefore, not applicable when an infinitesimal angle is under consideration. The following theorems apply to cases of the latter kind:

(ii) *If two great circles intersect at C, forming the infinitesimal angle α, their distance apart at a distance a from C is*

$$p = \alpha \sin a.$$

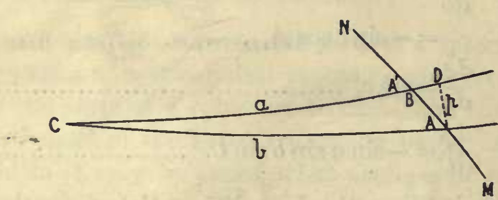


FIG. 1.

For, supposing $AD \equiv p$ to be a perpendicular to the arc CA, we have

$$\sin p = \sin \alpha \sin a,$$

from which the equation follows when α and p become infinitesimal.

(iii) *An arc CA cuts the transversal arc MN at the point A. If this arc turns on C through the infinitesimal angle α into the position CA′, the corresponding increment of the angle at A will be*

$$\Delta A = A' - A = \alpha \cos CA.$$

To apply the usual formulae of spherical trigonometry to this case we put B for the interior angle adjacent to A', and letter the remaining parts of the triangle ABC accordingly. Then

$$A' - A = 180° - (A + B),$$
$$\sin (A' - A) = \sin (A + B).$$

From the fundamental equation

$$\sin A \cos B + \cos A \sin B \cos c = \cos b \sin C,$$

we find, when c and C are infinitesimals,

$$\sin (A + B) = \cos b \sin C = \sin a \cos b.$$

In this case the part C reduces to α, and the part b to the arc CA. Hence, when C becomes infinitesimal we have the equation enunciated.

NOTES AND REFERENCES.

A prime requisite to the astronomical computer is a set of the most convenient logarithmic and other tables. The following are of this class:

Seven-place logarithms.

Bruhns, *New Manual of Logarithms*, Tauchnitz, Leipzig.

Zech, *Additions- und Subtractions-Logarithmen*, Hirzel, Leipzig.

The purpose of these last named tables is to find the logarithm of the sum or difference of two numbers given by their logarithms, without the labour of taking out the natural numbers and adding them.

Six-place logarithms.

Bremiker, *Logarithmisch-trigonometrische Tafeln*, Berlin, Nicolaische Verlags-Buchhandlung.

This edition contains also addition and subtraction logarithms.

Five-place logarithms.

Gauss, *Fünfstellige vollständige Logarithmische und Trigonometrische Tafeln*, Halle, Verlag von Eugen Strien.

Hussey, *Logarithmic and Other Mathematical Tables*, Ann Arbor, Mich., the Register Publishing Co.

Newcomb, *Five-place Logarithmic Tables*, New York, Henry Holt & Co.

Becker, *Logarithmisch-trigonometrisches Handbuch auf 5 Decimalen*, Tauchnitz, Leipzig.

The tables of Becker, of Gauss, and of Newcomb contain addition and subtraction logarithms, and other useful tables. The introduction to Newcomb's tables contains hints on the art of astronomical computation to facilitate the training of computers.

Four-place tables.

Gauss, *Vierstellige Logarithmisch-trigonometrische Handtafeln*.

Three-place tables.

NEWCOMB, 3- and 4-*place Logarithmic and Trigonometric Tables*, Appendix to Elements of Trigonometry, New York, Henry Holt & Co.; also separately, Washington, Lowdermilk & Co.

Tables of special kinds.

Besides the above tables which may be described as in regular form, the following are useful for special purposes:

GRAVELIUS, *Fünfstellige Logarithmisch-trigonometrische Tafeln*, Berlin, Reimer.

In these tables the degree is not divided into minutes and seconds but into hundredths, which is more convenient when a translation into minutes and seconds is not necessary.

For numbers with not more than three digits, when the logarithm of the product is not required, and when only two factors enter, a multiplication table may be used with more convenience than logarithms. The most extended set of tables of this sort is:

CRELLE-BREMIKER, *Rechentafeln*, Berlin, Reimer.

TAMBORREL, *Tablas de Multiplicar*, Mexico, Mariano Nava, is a much more compact book than Crelle's, and may serve the same purpose.

LECOY ET CLAUDEL, *Comptes faits*, Paris, gives all products of three figures by two.

Among miscellaneous tables for astronomical uses in general:

BAUSCHINGER, *Tafeln zur Theoretischen Astronomie*, is a well prepared and most useful work.

Astronomical ephemerides.

In the same class with the preceding may be placed the *National Astronomical Ephemerides*, published by the governments of the United States, Germany, England, France, and Spain under the respective titles:

The *American Ephemeris and Nautical Almanac*, Bureau of Equipment, Navy Department, Washington.

The *Nautical Almanac and Astronomical Ephemeris*, published by the order of the Lords Commissioners of the Admiralty, London.

Astronomisches Jahrbuch, Berlin.

Connaissance des Temps ou des Mouvements célestes, Paris, Bureau des Longitudes.

Almanaque Nautico, San Fernando, Spain.

These publications contain, for each year, tables of the varying quantities relating to the celestial motions. They are referred to collectively in the present work under the title of the *Astronomical Ephemeris*.

CHAPTER II.

OF DIFFERENCES, INTERPOLATION, AND DEVELOPMENT.

8. In astronomical tables and ephemerides the values of certain quantities are given for special equidistant epochs or values of an argument: say noon of every day, the beginning of every year, or every minute of the quadrant. When the values are required for an intermediate epoch, or value of the argument, the process of interpolation is necessary. There are various methods of applying this process, according to the greater or less complexity of the law of change of the tabular quantities to be found.

We first call to mind that, by the *rate of variation* of a quantity at a given moment, we mean the derivative of the quantity as to the time at that moment; that is to say, the change which the quantity would undergo in a unit of time if its rate of change remained constant. The unit may be a second, minute, hour, day or year, or even a century. The rate may be called the *variation* simply. The simplest cases of interpolation are the two following:

Case I. *The variation constant.* This is so simple a case as to need no explanation. To effect an interpolation we have only to multiply the variation by the elapsed time and add the product to the value given in the table. We must of course take care that the unit of time which we use corresponds to that for which the variation is given.

CASE II. *When the variation itself changes uniformly with the time.* We may treat this case by the infinitesimal calculus as follows:

Call u the given quantity, and let a and b be constants; the variation of u is, by hypothesis, of the form

$$\frac{du}{dt}=a+bt.$$

From this we derive by integration

$$\left.\begin{array}{l} u=u_0+at+\frac{1}{2}bt^2 \\ \text{or} \quad u=u_0+(a+\frac{1}{2}bt)t \end{array}\right\}, \quad \ldots\ldots\ldots(1)$$

u_0 being the value at the time from which t is reckoned.

The second form is commonly the most convenient one to use, and may be correctly arrived at in this way: By hypothesis we have

$$\text{At time } t=0;\ \text{variation}=a.$$

$$\text{,,} \quad t=t; \quad \text{,,} \quad =a+bt.$$

It will be seen that the variation which we found in formula (1) is the half sum of these variations, that is, the value corresponding to the middle of the interval over which we interpolated.

As an example, let us take from the Ephemeris the Right Ascension of the Moon at two consecutive hours of Greenwich mean time, on 1908, June 13, which we find to be as follows:

Hour.	R.A.	Variation for 1 minute.
	h. m. s.	s.
1	16 27 35·43	2·4280
2	16 30 1·31	2·4347

Let it be required to interpolate to the time 1 h. 36 m. Since the variation is constantly increasing, it is clear that if we used the variation at 1 h. the resulting difference would be too small; and if we used the variation at the time to which we interpolate, namely, 1 h. 36 m., it would be too great. What we must do is to take the variation at the moment which is half-way between the epochs, namely, 1 h. 18 m. or 1·3 h. This we find by simple interpolation to be 2·4280s. + 0·0067 s. × 0·3 = 2·4300 s. Multiplying this rate of change per minute by the 36 minutes which have elapsed, we find the interpolated value to be

$$\overset{\text{h.}}{16}\ \overset{\text{m.}}{27}\ \overset{\text{s.}}{35{\cdot}43} + \overset{\text{m.}}{1}\ \overset{\text{s.}}{27{\cdot}48} = \overset{\text{h.}}{16}\ \overset{\text{m.}}{29}\ \overset{\text{s.}}{2{\cdot}91}$$

In many tables and ephemerides, what is given is not the derivative, or variation per unit of time, but the difference between two consecutive values of the quantity, which is found by subtracting each value from the one which follows it. If we do this with the R.A.'s in the lunar ephemeris, we shall find that the differences vary nearly uniformly from hour to hour, and a little consideration will show that in this case they express the hourly variation at each half hour. The variation corresponding to the middle of the interval over which we interpolate may then be found by interpolating to this moment from the variations at the half hour preceding and following.

9. Differences of various orders.

As a general rule the quantities found in the ephemeris are given for intervals so short that the preceding methods of interpolation will suffice to give an accurate value of the quantity at any moment. But cases continually arise in astronomical practice in which the variation itself changes widely between two epochs. This more general case requires us to begin by pointing out the method of testing the accuracy of numbers by *successive differences*, defined as follows:

When we have several successive values of a varying quantity, the excess of each value over that preceding is called a *first difference*, or *difference of the first order*.

The excess of each first difference over the preceding one is called a *second difference*, or *difference of the second order*.

Continuing this process of subtraction we have third differences, fourth differences, etc.

The successive differences are generally designated by the symbols

$$\Delta', \ \Delta'', \ \Delta''', \ \Delta^{\text{iv}}, \ \ldots.$$

The best form for writing differences is that shown in the scheme (B) following. The first column contains only a series of indices which serve for the numbering of the individual differences. The next column gives the successive values of the function, which we call u. We then have

$$\left.\begin{array}{ll} \Delta'_{\frac{1}{2}} = u_1 - u_0, & \Delta''_1 = \Delta'_{\frac{3}{2}} - \Delta'_{\frac{1}{2}}, \text{ etc.} \\ \Delta'_{\frac{3}{2}} = u_2 - u_1, & \Delta''_2 = \Delta'_{\frac{5}{2}} - \Delta'_{\frac{3}{2}}, \text{ etc.} \\ \Delta'_{\frac{5}{2}} = u_3 - u_2, & \text{etc.} \\ \text{etc.} & \end{array}\right\} \ldots\ldots\ldots\ldots\ldots \text{(A)}$$

We arrange these quantities as follows:

$$\left.\begin{array}{ccccccc}
0 & u_0 & & & & & \\
 & — & \Delta'_{\frac{1}{2}} & & & & \\
1 & u_1 & — & \Delta''_1 & & & \\
 & — & \Delta'_{\frac{3}{2}} & — & \Delta'''_{\frac{3}{2}} & & \\
2 & u_2 & — & \Delta''_2 & — & \Delta^{\text{iv}}_2 & \\
 & — & \Delta'_{\frac{5}{2}} & — & \Delta'''_{\frac{5}{2}} & — & \Delta^{\text{v}}_{\frac{5}{2}} \\
3 & u_3 & — & \Delta''_3 & — & \Delta^{\text{iv}}_3 & — \\
 & — & \Delta'_{\frac{7}{2}} & — & \Delta'''_{\frac{7}{2}} & — & \Delta^{\text{v}}_{\frac{7}{2}} \\
4 & u_4 & — & \Delta''_4 & — & \Delta^{\text{iv}}_4 & — \\
 & — & \Delta'_{\frac{9}{2}} & — & \Delta'''_{\frac{9}{2}} & — & \Delta^{\text{v}}_{\frac{9}{2}} \\
5 & u_5 & — & \Delta''_5 & — & \Delta^{\text{iv}}_5 & — \\
 & — & \Delta'_{\frac{11}{2}} & — & \Delta'''_{\frac{11}{2}} & — & \Delta^{\text{v}}_{\frac{11}{2}} \\
6 & u_6 & — & \Delta''_6 & — & \Delta^{\text{iv}}_6 & \\
 & — & \Delta'_{\frac{13}{2}} & — & \Delta'''_{\frac{13}{2}} & & \\
7 & u_7 & — & \Delta''_7 & & & \\
 & — & \Delta'_{\frac{15}{2}} & & & & \\
8 & u_8 & & & & &
\end{array}\right\} \ldots\ldots\ldots\ldots\ldots \text{(B)}$$

It will be seen that each difference is written on the line between the two numbers to whose difference it is equal, and is distinguished by a suffix equal to half the sum of their suffixes. Thus, like suffixes are on a horizontal line; differences of even order all have integral suffixes; those of odd order fractional ones.

As an example we take the moon's longitude for Greenwich noon and midnight of the first few days of 1895 and difference it.

1895. Jan.	Longitude.	Δ′	Δ″	Δ‴	Δ^{iv}	Δ^{v}	
1·0	339° 36′ 53″·6						
		5° 55′ 50″·7					
1·5	345 32 44 ·3		1′ 55″·9				
		5 57 46 ·6		38″·4			
2·0	351 30 30 ·9		2 34 ·3		+2″·1		
		6 0 20 ·9		40 ·5		−2″·0	
2·5	357 30 51 ·8		3 14 ·8		+0 ·1		
		6 3 35 ·7		40 ·6		+0 ·5	(c)
3·0	3 34 27 ·5		3 55 ·4		+0 ·6		
		6 7 31 ·1		41 ·2		−1 ·9	
3·5	9 41 58 ·6		4 36 ·6		−1 ·3		
		6 12 7 ·7		39 ·9			
4·0	15 54 6 ·3		5 16 ·5				
		6 17 24 ·2					
4·5	22 11 30 ·5						

10. Detecting errors by differencing.

One of the most valuable applications of differencing is to the detection of isolated errors in the values of the functions whose differences are taken. Suppose that one of the values of u is affected by the error e, so that the table, instead of giving the value u, gives

$$u+e,$$

while all the other numbers are correct. We then readily conclude that the first differences before and after this quantity will be affected by the respective errors $+e$ and $-e$. The second differences will be affected by the errors $+e$, $-2e$, and $+e$. Continuing the process we find the resulting errors of the successive differences to be as follows:

u	of Δ'	of Δ''	of Δ'''	of Δ^{iv}	of Δ^{v}
0	—	0	—	0	—
—	0	—	0	—	$+e$
0	—	0	—	$+e$	—
—	0	—	$+e$	—	$-5e$
0	—	$+e$	—	$-4e$	—
—	$+e$	—	$-3e$	—	$+10e$
e	—	$-2e$	—	$+6e$	— ....(2)
—	$-e$	—	$+3e$	—	$-10e$
0	—	$+e$	—	$-4e$	—
—	0	—	$-e$	—	$+5e$
0	—	0	—	$+e$	—
—	0	—	0	—	$-e$
0	—	0	—	0	—

We see that an error in any one original quantity will be increased tenfold when carried out to the fifth difference, and can in all ordinary cases be detected, provided the adjacent quantities are correct.

The general expression for the coefficients of e in the errors of the nth differences is the same as that of the coefficients which enter into the binomial theorem, namely

$$\frac{n(n-1)(n-2)\dots(n-s+1)}{1\cdot 2\cdot 3\cdot\dots s},$$

where s takes the successive values 1, 2, 3, ... n. In applying this test it must be remembered that all the quantities we ordinarily obtain in astronomical computations are necessarily affected by the errors of the omitted decimals, which errors will shew themselves by the process of differencing.

How far it is necessary to carry the differences will depend upon the rapidity with which they converge. If the given numbers are mathematically exact, the differences will, if the quantities are given for values of the argument small enough to be used in interpolating, continually and rapidly diminish, so that, after a certain order, commonly not greater than the fifth or sixth, they become insensible. But the differences of the errors arising from omitted decimals will, as just shown, go on increasing with every order, and so will ultimately form the largest part of the column of differences. When this is the case the columns of differences will become irregular, the + and − signs generally alternating.

Other points to be borne in mind are these:

α. If errors are numerous but accidental, the fact that they exist will be shown by the differences, but it may be impossible to determine what numbers are wrong; whereas this is easy in the case of a single isolated error.

β. A systematic error, *i.e.*, one which runs through all the numbers and follows any law whatever, cannot be detected by differencing.

11. Use of higher orders of differences in interpolation.

There are two applications of the method of interpolation by differences.

(i) When, from several values of a variable quantity, given in tabular form, it is desired to find some intermediate value for one or more special values of the argument.

(ii) When it is desired to make a table in which the intervals shall be smaller than those of the quantities originally computed. For example, the position of a planet may be computed, in the first place, for every fourth, fifth, or tenth day; and it may then be required to form an ephemeris for all the omitted days by interpolation. In the second application the same general formulae are used as in the first; we shall therefore give a brief summary of them.

First application. As before, we put u for the variable quantity for which we have computed the values for a number of equidistant epochs; we suppose the successive differences of u formed so far as necessary, and we call

$$u_0 \text{ and } u_1$$

the two consecutive values of u between which we wish to interpolate a new value.

It is a known theorem of algebra that the nth value of u following u_0, which is in fact u_n, is given in terms of u_0, and of the successive differences in a diagonal line descending from u_0 by the equation

$$u_n = u_0 + n\Delta'_{\frac{1}{2}} + \frac{n(n-1)}{1.2}\Delta''_1 + \frac{n(n-1)(n-2)}{1.2.3}\Delta'''_{\frac{3}{2}} + \ldots, \ldots\ldots(3)$$

the coefficients being those of the binomial theorem.

The fundamental hypothesis of interpolation is that this equation, which is rigorous only when n is a positive integer, will also give the value of u when n is a fraction. This hypothesis, though not necessarily true, and failing entirely in cases when the law of change in u is not shown by the differences, is quite safe in practice if we compute the values of u for intervals so small that the orders of differences are convergent *both for these and for all shorter intervals.* Let us put

T_0; T_1; the two times for which the values u_0 and u_1, between which we interpolate a new value, are computed.

T, the time for which we wish to interpolate. Then

$$\frac{T-T_0}{T_1-T_0} = t$$

will be the time after T_0, expressed in terms of the interval of computation as the unit. For example, if this interval were two days, and we wished to find the value of u for a moment 8 hours after one of the times of computation, we should put

$$t = 8 \div 48 = 0{\cdot}1666 \ldots.$$

Evidently when t is an integer it will correspond to n, as already defined, so that the equation (3) becomes

$$u_t = u_0 + t\Delta'_{\frac{1}{2}} + \frac{t(t-1)}{1.2}\Delta''_1 + \frac{t(t-1)(t-2)}{1.2.3}\Delta'''_{\frac{3}{2}} + \text{etc.} \ldots\ldots\ldots(4)$$

This is Newton's formula of interpolation, and forms the basis of all the other formulae in common use.

12. Transformations of the formula of interpolation.

In Newton's formula each successive difference which enters is taken half a line below the preceding one. The series is, however, more convergent when the differences of alternate orders are taken on the same horizontal line. The transformations to effect this will now be shown.

In all cases in which it is worth while to interpolate, the differences beyond the fifth will not affect the result. We shall therefore suppose the differences of the sixth and all following orders to vanish, which amounts to supposing those of the fifth order constant.

We first consider the form in which the original differences on alternate lines are used. Let us express u_t in terms of the quantities shown in the following scheme:

$$\left.\begin{matrix} u_0 & — & \Delta''_0 & — & \Delta^{iv}_0 & — \\ — & \Delta'_{\frac{1}{2}} & — & \Delta'''_{\frac{1}{2}} & — & \Delta^{v}_{\frac{1}{2}} \end{matrix}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (b)$$

We express the differences in (4) in terms of these as follows:

$$\begin{aligned} \Delta''_1 &= \Delta''_0 + \Delta'''_{\frac{1}{2}} \\ \Delta'''_{\frac{3}{2}} &= \Delta'''_{\frac{1}{2}} + \Delta^{iv}_1 \\ &= \Delta'''_{\frac{1}{2}} + \Delta^{iv}_0 + \Delta^{v}_{\frac{1}{2}} \\ \Delta^{iv}_2 &= \Delta^{iv}_0 + 2\Delta^{v}_{\frac{1}{2}} + \Delta^{vi}_1 \\ \Delta^{v}_{\frac{5}{2}} &= \Delta^{v}_{\frac{1}{2}} + 2\Delta^{vi}_1 + \Delta^{vii}_{\frac{3}{2}} \end{aligned}$$

Making these substitutions in (4) and putting $\Delta^{vi} = \Delta^{vii} = 0$, we find, by reducing and collecting the coefficients of the differences in (b),

$$\left.\begin{aligned} u_t - u_0 &= t\Delta'_{\frac{1}{2}} + \frac{t(t-1)}{2}\Delta''_0 + \frac{(t+1)t(t-1)}{1.2.3}\Delta'''_{\frac{1}{2}} \\ &\quad + \frac{(t+1)t(t-1)(t-2)}{1.2.3.4}\Delta^{iv}_0 + \frac{(t+2)(t+1)t(t-1)(t-2)}{1.2.3.4.5}\Delta^{v}_{\frac{1}{2}}, \\ \text{or } u_t - u_0 &= t\Delta'_{\frac{1}{2}} + \frac{t(t-1)}{1.2}\Delta''_0 + \frac{t(t^2-1)}{1.2.3}\Delta'''_{\frac{1}{2}} + \frac{t(t^2-1)(t-2)}{1.2.3.4}\Delta^{iv}_0 \\ &\quad + \frac{t(t^2-1)(t^2-4)}{1.2.3.4.5}\Delta^{v}_{\frac{1}{2}}. \end{aligned}\right\} (5)$$

Let us next take the differences of odd orders one line higher, thus:

$$\left.\begin{matrix} — & \Delta'_{-\frac{1}{2}} & — & \Delta'''_{-\frac{1}{2}} & — & \Delta^{v}_{-\frac{1}{2}} \\ u_0 & — & \Delta''_0 & — & \Delta^{iv}_0 & — \end{matrix}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (c)$$

We have

$$\begin{aligned} \Delta'_{\frac{1}{2}} &= \Delta'_{-\frac{1}{2}} + \Delta''_0, \\ \Delta'''_{\frac{1}{2}} &= \Delta'''_{-\frac{1}{2}} + \Delta^{iv}_0, \\ \Delta^{v}_{\frac{1}{2}} &= \Delta^{v}_{-\frac{1}{2}} + \Delta^{vi}_0 = \Delta^{v}_{-\frac{1}{2}}. \end{aligned}$$

Making these substitutions in (5), we find

$$\left.\begin{aligned}u_t-u_0=t\Delta'_{-\frac{1}{2}}+\frac{t(t+1)}{1.2}\Delta''_0+\frac{t(t^2-1)}{1.2.3}\Delta'''_{-\frac{1}{2}}+\frac{t(t^2-1)(t+2)}{1.2.3.4}\Delta^{\text{iv}}_0\\+\frac{t(t^2-1)(t^2-4)}{1.2.3.4.5}\Delta^{\text{v}}_{-\frac{1}{2}}\end{aligned}\right\}. \quad (6)$$

Now let us make a third transformation by using the differences next below those of (c), thus:

$$\left.\begin{matrix}u_0 & & & & & & \\ - & \Delta'_{\frac{1}{2}} & - & \Delta'''_{\frac{1}{2}} & - & \Delta^{\text{v}}_{\frac{1}{2}} \\ u_1 & - & \Delta''_1 & - & \Delta^{\text{iv}}_1 & - \end{matrix}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (d)$$

If, in (6), we substitute these differences for those there written, we shall have the value of $u_{t+1}-u_1$, that is, putting

$$t'=t+1,$$

we shall have the value of $u_{t'}-u_1=u_{t'}-u_0-\Delta'_{\frac{1}{2}}$. Substituting in (6) for t its value $t'-1$ and the differences (d), and then dropping the accent, we find

$$\left.\begin{aligned}u_t-u_0=t\Delta'_{\frac{1}{2}}+\frac{t(t-1)}{1.2}\Delta''_1+\frac{t(t-1)(t-2)}{1.2.3}\Delta'''_{\frac{1}{2}}+\frac{t(t^2-1)(t-2)}{1.2.3.4}\Delta^{\text{iv}}_1\\+\frac{t(t^2-1)(t-2)(t-3)}{1.2.3.4.5}\Delta^{\text{v}}_{\frac{1}{2}}\end{aligned}\right\}. \quad (7)$$

The formulae (5), (6), and (7) have the advantage that the differences of each alternate order are taken from the same horizontal line. But a yet farther transformation is necessary to reduce the equations to the best form for practical use.

Referring to the scheme (B) it will be seen that there are no values of Δ', Δ''', etc., with entire suffixes, and no values of Δ'', Δ^{iv}, etc., with fractional suffixes, but that the places where these values might go are left blank. Now, we may imagine that these blanks are filled by quantities given by the general equation

$$\Delta_{i+\frac{1}{2}}=\tfrac{1}{2}(\Delta_i+\Delta_{i+1}); \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

that is, in each blank space we may imagine written the half sum of the Δ's above and below it, and we designate these half sums by Δ with half the sum of the suffixes attached. We use this notation in what follows.

13. Stirling's formula of interpolation.

In this formula the following scheme is used:

$$u_0,\ \Delta_0',\ \Delta_0'',\ \Delta_0''',\ \Delta_0^{iv},\ \Delta_0^{v},\ \text{etc.}$$

Δ_0', Δ_0''', Δ_0^{v}, etc., being defined as just described. We derive the formula by taking the half sum of (5) and (6), which gives

$$\left.\begin{aligned} u_t = u_0 + t\Delta_0' + \frac{t^2}{1.2}\Delta_0'' + \frac{t(t^2-1)}{1.2.3}\Delta_0''' \\ + \frac{t^2(t^2-1)}{1.2.3.4}\Delta_0^{iv} + \frac{t(t^2-1)(t^2-4)}{1.2.3.4.5}\Delta_0^{v} + \text{etc.} \end{aligned}\right\}, \quad \ldots(9)$$

which may be used equally well for either positive or negative values of t.

14. Bessel's formula of interpolation.

The scheme of Δ's used in this formula is

$$\left.\begin{matrix} u_0 & - & - & - & - & - \\ - & \Delta_{\frac{1}{2}}' & \Delta_{\frac{1}{2}}'' & \Delta_{\frac{1}{2}}''' & \Delta_{\frac{1}{2}}^{iv} & \Delta_{\frac{1}{2}}^{v} \\ u_1 & - & - & - & - & - \end{matrix}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(e)$$

The formula is found by taking the half sum of (5) and (7) which is:

$$\left.\begin{aligned} u_t - u_0 = t\Delta_{\frac{1}{2}}' + \frac{t(t-1)}{1.2}\Delta_{\frac{1}{2}}'' + \frac{t(t-1)(t-\frac{1}{2})}{1.2.3}\Delta_{\frac{1}{2}}''' \\ + \frac{t(t^2-1)(t-2)}{1.2.3.4}\Delta_{\frac{1}{2}}^{iv} + \frac{t(t^2-1)(t-2)(t-\frac{1}{2})}{1.2.3.4.5}\Delta_{\frac{1}{2}}^{v} \end{aligned}\right\}, \quad (10)$$

which is Bessel's formula in its usual shape.

The second member of this is less simple than that of Stirling's formula, where the differences of odd order have as coefficients only odd powers of t, and those of even order only even powers. But we may make it more symmetrical by filling the blank between u_0 and u_1 in the scheme (e) by $\frac{1}{2}(u_0+u_1) \equiv u_{\frac{1}{2}}'$, and counting t (considered as the time) from the corresponding moment, which we may do by putting

$$t' = t - \tfrac{1}{2}; \quad \therefore\ t = t' + \tfrac{1}{2}.$$

First we shall have for substitution in (10)

$$u_0 = u_{\frac{1}{2}}' - \tfrac{1}{2}\Delta_{\frac{1}{2}}'.$$

$$u_t = u_{\frac{1}{2}+t'};$$

$$\therefore\ u_t - u_0 = u_{\frac{1}{2}+t'} - u_{\frac{1}{2}}' + \tfrac{1}{2}\Delta_{\frac{1}{2}}'. \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Then by substitution in (10)

$$\left.\begin{aligned} u_{\frac{1}{2}+t'} = u'_{\frac{1}{2}} + t'\Delta'_{\frac{1}{2}} + \frac{t'^2-\frac{1}{4}}{1.2}\Delta''_{\frac{1}{2}} + \frac{t'(t'^2-\frac{1}{4})}{1.2.3}\Delta'''_{\frac{1}{2}} \\ + \frac{(t'^2-\frac{1}{4})(t'^2-\frac{9}{4})}{1.2.3.4}\Delta^{\text{iv}}_{\frac{1}{2}} + \frac{t'(t'^2-\frac{1}{4})(t'^2-\frac{9}{4})}{1.2.3.4.5}\Delta^{\text{v}}_{\frac{1}{2}} + \text{etc.} \end{aligned}\right\}, \quad (12)$$

which is now as symmetrical as Stirling's formula.

When one or more isolated values are to be interpolated, either of the formulae (8), (9), or (10) may be used with nearly equal convenience. In practice it will often be convenient to use (7), factoring it thus:

$$u_t = u_0 + t\left[\Delta'_{\frac{1}{2}} + \frac{t-1}{2}\left\{\Delta''_1 + \frac{t-2}{3}(\Delta'''_{\frac{1}{2}} + \frac{t+1}{4}(\Delta^{\text{iv}}_1 + \ldots))\right\}\right]. \quad (13)$$

The most common application of interpolation is the second described in § 11. Such an interpolation is said to be to halves, to thirds, to fifths, etc., according as the new intervals are one half, one third, one fifth, etc., of the original ones. The most expeditious way of executing such an interpolation is to compute the first differences of the interpolated series, and then find the required quantities in succession by adding these differences.

The following examples of the way of practically executing the work are mostly from the Introduction to the author's tables of five-place logarithms.

15. Interpolation to halves.

It is required, from the logarithms of 310, 320, 330 ... 360 to find those of 315, 325 ... 355.

Here the required quantities depending upon arguments half way between the given ones; we have $t=\frac{1}{2}$, and the values of the Besselian coefficients, so far as wanted, are

$$\frac{t(t-1)}{2} = -\frac{1}{8},$$

$$\frac{t(t-1)(t-\frac{1}{2})}{6} = 0,$$

$$\frac{t(t^2-1)(t-2)}{24} = \frac{3}{128}.$$

The subsequent terms are neglected, being insensible. So, if we put a_0 and a_1 for any consecutive two of the numbers 300, 310, etc., we have

$$\left.\begin{aligned}\log(a_0+5)&=\log a_0+\left(\frac{1}{2}\Delta'_{\frac{1}{2}}-\frac{1}{8}\frac{\Delta''_0+\Delta''_1}{2}+\frac{3}{128}\frac{\Delta^{iv}_0+\Delta^{iv}_1}{2}\right)\\ \log(a_1-5)&=\log a_1-\left(\frac{1}{2}\Delta'_{\frac{1}{2}}+\frac{1}{8}\frac{\Delta''_0+\Delta''_1}{2}-\frac{3}{128}\frac{\Delta^{iv}_0+\Delta^{iv}_1}{2}\right)\end{aligned}\right\}, \quad (14)$$

where we put $\Delta'_{\frac{1}{2}}$ for the difference between $\log a_0$ and $\log a_1$.

These two formulae are two expressions for the same quantity, because $a_0+5=a_1-5$. They are both used in such a way as to provide a check upon the accuracy of the work. For this purpose we compute the two quantities

$$\left.\begin{aligned}\log(a_0+5)-\log a_0&=\frac{1}{2}\Delta'_{\frac{1}{2}}-\frac{1}{8}\frac{\Delta''_0+\Delta''_1}{2}+\text{etc.}\\ \log a_1-\log(a_0+5)&=\frac{1}{2}\Delta'_{\frac{1}{2}}+\frac{1}{8}\frac{\Delta''_0+\Delta''_1}{2}+\text{etc.}\end{aligned}\right\}. \quad \dots(15)$$

The most convenient and expeditious way of doing the work is shown in the accompanying table, where we give every figure which it is necessary to write. The letters (*a*), (*b*) ... (*i*) at the tops of the columns show the order in which the columns are written.

(*a*)	(*b*) (*i*)	(*h*)	(*g*)	(*f*)	(*e*)	(*c*)	(*d*)
No.	Log.	Diff.	$\frac{1}{2}\Delta'_{\frac{1}{2}}$	$\frac{1}{8}\frac{\Delta''_0+\Delta''_1}{2}$	$\frac{\Delta''_0+\Delta''_1}{2}$	Δ'	Δ''
310	2·491 36						
		695					
315	·498 31		689·5	−5·5	−44	1379	
		684					
320	·505 15						−43
		673					
325	·511 88		668·0	−5·1	−41	1336	
		663					
330	·518 51						−39
		653					
335	·525 04		648·5	−4·8	−38	1297	
		644					

(a)	(b) (i)	(h)	(g)	(f)	(e)	(c)	(d)
No.	Log.	Diff.	$\frac{1}{2}\Delta'_{\frac{1}{2}}$	$\frac{1}{8}\frac{\Delta''_0+\Delta''_1}{2}$	$\frac{\Delta''_0+\Delta''_1}{2}$	Δ'	Δ''
340	**·531 48**						**−38**
		634					
345	·537 82		629·5	−4·6	−37	**1259**	
		625					
350	**·544 07**						**−36**
		616					
355	·550 23		611·5	−4·3	−34	**1223**	
		607					
360	**2·556 30**						

We compute the column (e) by the formula

$$\frac{\Delta''_0+\Delta''_1}{2}=\Delta''_0+\frac{1}{2}\Delta'''_{\frac{1}{2}}=\Delta''_1-\frac{1}{2}\Delta'''_{\frac{1}{2}},$$

the set of suffixes 0, 1 and $\frac{1}{2}$ being applied in succession to each set of differences which enter into the computation.

This mode of computing the half sum of two numbers which are nearly equal is easier than adding and dividing by 2.

In the next two columns to the left, the sixth place of decimals is added in order that the errors may not accumulate by addition. This precaution should always be taken when the interpolated quantities are required to be as accurate as the given ones.

The fourth column from the right is formed by adding and subtracting the numbers of the second and third columns according to the formula (15). The additional figure is now dropped, because no longer necessary for accuracy. The numbers thus formed are the first differences of the series of logarithms between the given ones, as will be seen by equation (15).

We write the first logarithm of the series, namely,

$$\log 310 = 2{\cdot}491\ 36,$$

and then form the subsequent ones by continual addition of the differences, thus:

$$\log 315=\log 310+695;$$
$$\log 320=\log 315+684;$$
$$\log 325=\log 320+673;$$
etc., etc., etc.

If the work is correct, the alternate logarithms will agree with the given ones in the former table.

The continuance of the above process for a few more numbers, say up to 450, is recommended to the student as an exercise.

16. Interpolation to thirds.

Let the value of a quantity be given for every third day, and the value for every day be required. By putting $t=\frac{1}{3}$ and applying Bessel's formula to each successive given quantity, we shall have the value for each day following one of those given, and by putting $t=\frac{2}{3}$ we shall have values for the second day following, which will complete the series. But the interpolation can be executed by a much more expeditious process, which consists in computing the middle difference of the interpolated quantities and finding the intermediate differences by a secondary interpolation.

Let us put

f_0, f_3, f_6, etc., the given series of quantities;

f_0, f_1, f_2, f_3, f_4, etc., the required interpolated series;

Δ', Δ'', etc., the first differences, second differences, etc., of the given series;

δ', δ'', etc., the first differences second differences, etc., of the interpolated series.

We may then put

$$f_3-f_0=\Delta'_{\frac{1}{2}} \quad \text{(in the given series);}$$

$$\left.\begin{array}{l} f_1-f_0=\delta'_{\frac{1}{2}} \\ f_2-f_1=\delta'_{\frac{3}{2}} \\ f_3-f_2=\delta'_{\frac{5}{2}} \end{array}\right\} \text{(in the interpolated series).}$$

We shall then have $\delta'_{\frac{1}{2}}+\delta'_{\frac{3}{2}}+\delta'_{\frac{5}{2}}=\Delta'_{\frac{1}{2}}$.

The value of $f_1-f_0=\delta'_{\frac{1}{2}}$ is given by putting $t=\frac{1}{3}$ in the Besselian formula (10). Thus we find

$$\delta'_{\frac{1}{2}}=\frac{1}{3}\Delta'_{\frac{1}{2}}-\frac{1}{9}\,\frac{\Delta''_0+\Delta''_1}{2}+\frac{1}{162}\Delta'''_{\frac{1}{2}}+\frac{5}{243}\,\frac{\Delta^{\text{iv}}_0+\Delta^{\text{iv}}_1}{2}-\frac{1}{1458}\Delta^{\text{v}}_{\frac{1}{2}}.$$

Putting $t=\frac{2}{3}$, we have the value of f_2-f_0, that is, of $\delta'_{\frac{1}{2}}+\delta'_{\frac{3}{2}}$. Thus we find

$$\delta'_{\frac{1}{2}}+\delta'_{\frac{3}{2}}=\frac{2}{3}\Delta'_{\frac{1}{2}}-\frac{1}{9}\frac{\Delta''_0+\Delta''_1}{2}-\frac{1}{162}\Delta'''_{\frac{1}{2}}+\frac{5}{243}\frac{\Delta^{iv}_0+\Delta^{iv}_1}{2}+\frac{1}{1458}\Delta^{v}_{\frac{1}{2}}.$$

Subtracting these expressions from each other, we have

$$\delta'_{\frac{3}{2}}=\frac{1}{3}\Delta'_{\frac{1}{2}}-\frac{1}{81}\Delta'''_{\frac{1}{2}}+\frac{1}{729}\Delta^{v}_{\frac{1}{2}},$$

which is easily computed in the form

$$\delta'_{\frac{3}{2}}=\frac{1}{3}\left\{\Delta'_{\frac{1}{2}}-\frac{1}{27}\left(\Delta'''_{\frac{1}{2}}-\frac{1}{9}\Delta^{v}_{\frac{1}{2}}\right)\right\}. \quad\ldots\ldots\ldots\ldots\ldots(16)$$

We see that the computation of $\delta'_{\frac{3}{2}}$, the middle difference of the interpolated quantities, is much simpler than that of $\delta'_{\frac{1}{2}}$. It will therefore facilitate the work to compute only these middle differences, and to find the others by interpolation.

This process is again facilitated, in case the second differences are considerable, by first computing the second differences of the interpolated series on the same plan. The formulae for this purpose are derived as follows:

Let us put $\delta'_{\frac{7}{2}}=f_4-f_3.$

The second difference of which we desire the value is then

$$\delta''_3=\delta'_{\frac{7}{2}}-\delta'_{\frac{5}{2}}.$$

The value of $\delta'_{\frac{5}{2}}$ is given by the equation

$$\delta'_{\frac{5}{2}}=\Delta'_{\frac{1}{2}}-(\delta'_{\frac{1}{2}}+\delta'_{\frac{3}{2}}),$$

and the value of $\delta'_{\frac{7}{2}}$ is found from that of $\delta'_{\frac{1}{2}}$ by simply increasing the indices of the differences by unity, because it belongs to the next lower line.

We thus find

$$\delta'_{\frac{7}{2}}=\frac{1}{3}\Delta'_{\frac{3}{2}}-\frac{1}{9}\frac{\Delta''_1+\Delta''_2}{2}+\frac{1}{162}\Delta'''_{\frac{3}{2}}+\frac{5}{243}\frac{\Delta^{iv}_1+\Delta^{iv}_2}{2}-\frac{1}{1458}\Delta^{v}_{\frac{3}{2}};$$

$$\delta'_{\frac{5}{2}}=\frac{1}{3}\Delta'_{\frac{1}{2}}+\frac{1}{9}\frac{\Delta''_0+\Delta''_1}{2}+\frac{1}{162}\Delta'''_{\frac{1}{2}}-\frac{5}{243}\frac{\Delta^{iv}_0+\Delta^{iv}_1}{2}-\frac{1}{1458}\Delta^{v}_{\frac{1}{2}}.$$

Then by subtraction,

$$\delta_3'' = \frac{1}{3}(\Delta'_{\frac{3}{2}} - \Delta'_{\frac{1}{2}}) - \frac{1}{9}\,\frac{\Delta''_0 + 2\Delta''_1 + \Delta''_2}{2} + \frac{1}{162}(\Delta'''_{\frac{3}{2}} - \Delta'''_{\frac{1}{2}})$$

$$+ \frac{5}{243}\,\frac{\Delta^{iv}_0 + 2\Delta^{iv}_1 + \Delta^{iv}_2}{2} - \frac{1}{1458}(\Delta^{v}_{\frac{3}{2}} - \Delta^{v}_{\frac{1}{2}}).$$

Reducing the first of these terms, we have

$$\Delta'_{\frac{3}{2}} - \Delta'_{\frac{1}{2}} = \Delta''_1.$$

For the second term, $\Delta''_0 = \Delta''_1 - \Delta'''_{\frac{1}{2}}$;

$$\Delta''_2 = \Delta''_1 + \Delta'''_{\frac{3}{2}};$$

whence $\Delta''_0 + \Delta''_2 = 2\Delta''_1 + \Delta'''_{\frac{3}{2}} - \Delta'''_{\frac{1}{2}} = 2\Delta''_1 + \Delta^{iv}_1,$

and $$\frac{\Delta''_0 + 2\Delta''_1 + \Delta''_2}{2} = 2\Delta''_1 + \frac{1}{2}\Delta^{iv}_1.$$

For the third term, $\Delta'''_{\frac{3}{2}} - \Delta'''_{\frac{1}{2}} = \Delta^{iv}_1.$

For the fourth term, dropping the terms in Δ^{vi} as too small in practice, we may put

$$\frac{\Delta^{iv}_0 + 2\Delta^{iv}_1 + \Delta^{iv}_2}{2} = 2\Delta^{iv}_1.$$

The difference of the fifth terms may also be dropped, because they contain only sixth differences.

Making these substitutions in the value of δ_3'', we find

$$\left.\begin{aligned} \delta_3'' &= \frac{1}{3}\Delta''_1 - \frac{1}{9}\left(2\Delta''_1 + \frac{1}{2}\Delta^{iv}_1\right) + \frac{1}{162}\Delta^{iv}_1 + \frac{10}{243}\Delta^{iv}_1 \\ &= \frac{1}{9}\Delta''_1 - \frac{2}{243}\Delta^{iv}_1 \\ &= \frac{1}{9}\left(\Delta''_1 - \frac{2}{27}\Delta^{iv}\right). \end{aligned}\right\} \quad \text{.......(17)}$$

By this formula we may compute every third value of δ'', and then interpolate the intermediate values. By means of these values we find by addition the intermediate values of δ', of which every third value has been computed by formula (16). Then, by continually adding the values of δ', we find those of the function f.

As an example of the work, we give the following values of the sun's declination for every third day of part of July 1886, for Greenwich mean noon:

Date. 1886.	☉'s Dec. ° ′ ″	Δ′ ′ ″	Δ″ ″	Δ‴ ″
July 3	22 57 37·5			
		−16 28·3		
6	22 41 9·2		−212·4	
		−20 0·7		+4·5
9	22 21 8·5		−207·9	
		−23 28·6		+4·5
12	21 57 39·9		−203·4	
		−26 52·0		+5·7
15	21 30 47·9		−197·7	
		−30 9·7		
18	21 0 38·2			

The values of Δ^{iv} are too small to have any influence.

The whole work of interpolation is shown in the following table, where, as before, the right-hand column is that first computed, and gives the value of $\Delta' - \frac{1}{27}\Delta'''$ according to formula (16):

Date. 1886.	☉'s Dec. ° ′ ″	δ′ ′ ″	δ″ ″	$\Delta' - \frac{1}{27}\Delta'''$
July **6**	**22 41 9·2**		**−23·60**	
		−6 16·86		
7	22 34 52·4		−23·43	
		−6 40·29		−20 0·87
8	22 28 12·1		−23·27	
		−7 3·56		
9	**22 21 8·5**		**−23·10**	
		−7 26·66		
10	22 13 41·9		−22·93	
		−7 49·59		−23 28·77
11	22 5 52·3		−22·78	
		−8 12·37		
12	**21 57 39·9**		**−22·61**	
		−8 34·98		
13	21 49 4·9		−22·42	
		−8 57·40		−26 52·21
14	21 40 7·5		−23·19	
		−9 19·59		
15	**21 30 47·9**		**−21·97**	

To make the process in the example clear, the computed differences, etc., are printed in heavier type than the interpolated ones.

It is also to be remarked that the sum of the three consecutive values of δ'', formed of one computed value and the interpolated values next above and below it, should be equal to the difference between the corresponding computed first differences. For instance,

$$23''\cdot 27 + 23''\cdot 10 + 22''\cdot 93 = 7'\ 49''\cdot 59 - 6'\ 40''\cdot 29.$$

But in the first computation this condition will seldom be exactly fulfilled, owing to the errors arising from omitted decimals and other sources. If the given quantities are accurate, the errors should never exceed half a unit of the last decimal in the given quantities, or five units in the additional decimal added on in dividing.

To correct these little imperfections after the interpolation of the second differences, but before that of the first differences, the sum of the last two figures in each triplet of second differences should be formed, and if it does not agree with the difference of the first differences, the last figures of the second difference should each be slightly altered, to make the sum exact.

The first difference can then be formed by addition.

In the same way, the sum of three consecutive first differences should be equal to the difference between the given quantities. If, as is generally the case, this condition is not exactly fulfilled, the differences should be altered accordingly. This alteration may, however, be made mentally while adding to form the required interpolated functions.

17. Interpolation to fourths.

This may be effected by two successive interpolations to halves. The processes may be combined thus:

Let us put

$$\delta_1;\ \delta_2;\ \delta_3;\ \delta_4;$$

the four first differences of the interpolated series, so that

$$\delta_1 + \delta_2 + \delta_3 + \delta_4 = \Delta'_{\frac{1}{2}}.$$

Then, by (14), we have

$$\delta_2+\delta_1=\frac{1}{2}\Delta'_{\frac{1}{2}}-\frac{1}{8}\,\frac{\Delta''_0+\Delta''_1}{2}+\frac{3}{128}\,\frac{\Delta^{iv}_0+\Delta^{iv}_1}{2},$$

$$\delta_4+\delta_3=\frac{1}{2}\Delta'_{\frac{1}{2}}+\frac{1}{8}\,\frac{\Delta''_0+\Delta''_1}{2}-\frac{3}{128}\,\frac{\Delta''_0+\Delta''_1}{2}.$$

From Bessel's formula, by putting $t=\frac{1}{4}, \frac{2}{4}, \frac{3}{4}$ in succession we find

$$\delta_2-\delta_1=\frac{1}{16}\,\frac{\Delta''_0+\Delta''_1}{2}-\frac{1}{64}\Delta'''_{\frac{1}{2}}-\frac{11}{1024}\,\frac{\Delta^{iv}_0+\Delta^{iv}_1}{2},$$

$$\delta_4-\delta_3=\frac{1}{16}\,\frac{\Delta''_0+\Delta''_1}{2}+\frac{1}{64}\Delta'''_{\frac{1}{2}}-\frac{11}{1024}\,\frac{\Delta^{iv}_0+\Delta^{iv}_1}{2}.$$

In practice we first compute

$$\tfrac{1}{2}(\delta_2+\delta_1)=\tfrac{1}{4}\Delta'_{\frac{1}{2}}-\tfrac{1}{16}\Delta''_{\frac{1}{2}}+\tfrac{3}{256}\Delta^{iv}_{\frac{1}{2}}\equiv(1),$$

$$\tfrac{1}{2}(\delta_2-\delta_1)=\tfrac{1}{32}\Delta''_{\frac{1}{2}}-\tfrac{1}{128}\Delta'''_{\frac{1}{2}}-\tfrac{11}{2048}\Delta^{iv}_{\frac{1}{2}}\equiv(2),$$

$$\tfrac{1}{2}(\delta_4+\delta_3)=\tfrac{1}{4}\Delta'_{\frac{1}{2}}+\tfrac{1}{16}\Delta''_{\frac{1}{2}}-\tfrac{3}{256}\Delta^{iv}_{\frac{1}{2}}\equiv(3),$$

$$\tfrac{1}{2}(\delta_4-\delta_3)=\tfrac{1}{32}\Delta''_{\frac{1}{2}}+\tfrac{1}{128}\Delta'''_{\frac{1}{2}}-\tfrac{11}{2048}\Delta^{iv}_{\frac{1}{2}}\equiv(4),$$

and then

$$\left.\begin{aligned}\delta_1&=(1)-(2),\\ \delta_2&=(1)+(2),\\ \delta_3&=(3)-(4),\\ \delta_4&=(3)+(4).\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18)$$

18. Interpolation to fifths.

Let us next investigate the formulae when every fifth quantity is given and the intermediate ones are to be found by interpolation. By putting $t=\frac{2}{5}$ in the Besselian formula, we shall have the value of the interpolated function second following one of the given ones, and by putting $t=\frac{3}{5}$ that third following. The difference will be the middle interpolated first difference of the interpolated series.

Putting $t=\frac{2}{5}$ in (10), we have

$$u_{\frac{2}{5}}=u_0+\frac{2}{5}\Delta'_{\frac{1}{2}}-\frac{2.3}{2.5}\,\frac{\Delta''_0+\Delta''_1}{2}+\frac{2.3.1}{2^2.3.5^3}\Delta'''_{\frac{1}{2}}+\frac{2.3.7.8}{2.3.4.5^4}\,\frac{\Delta^{iv}_0+\Delta^{iv}_1}{2}$$
$$-\frac{2.3.7.8.1}{2^2.3.4.5.5^5}\Delta^{v}_{\frac{1}{2}}.$$

Putting $t=\frac{3}{5}$, we have

$$u_{\frac{3}{5}}=u_0+\frac{3}{5}\Delta'_{\frac{1}{2}}-\frac{2.3}{2.5^5}\frac{\Delta''_0+\Delta''_1}{2}-\frac{2.3.1}{2^2.3.5^3}\Delta'''_{\frac{1}{2}}+\frac{2.3.7.8}{2.3.4.5^4}\frac{\Delta^{iv}_0+\Delta^{iv}_1}{2}$$
$$+\frac{2.3.7.8.1}{2^2.3.4.5.5^5}\Delta^{v}_{\frac{1}{2}}.$$

The difference of these expressions, being reduced, gives

$$u_{\frac{3}{5}}-u_{\frac{2}{5}}=\frac{1}{5}\Delta'_{\frac{1}{2}}-\frac{1}{125}\Delta'''_{\frac{1}{2}}+\frac{14}{15625}\Delta^{v}_{\frac{1}{2}}$$
$$=\frac{1}{5}\left\{\Delta'_{\frac{1}{2}}-\frac{1}{25}\left(\Delta'''_{\frac{1}{2}}-\frac{14}{125}\Delta^{v}_{\frac{1}{2}}\right)\right\}.$$

The term in Δ^{v} will not produce any effect unless the fifth differences are considerable, and then we may nearly always, in practice, put $\frac{1}{9}$ instead of $\frac{14}{125}$.

The interpolated second differences opposite the given functions are most readily obtained by Stirling's formula (9). Putting $t=\frac{1}{5}$, we have the following value of the interpolated first differences immediately following a given value of the function:

$$u_{\frac{1}{5}}-u_0=\frac{1}{5}\frac{\Delta'_{-\frac{1}{2}}+\Delta'_{\frac{1}{2}}}{2}+\frac{1}{50}\Delta''_0-\frac{24}{6.5.25}\frac{\Delta'''_{-\frac{1}{2}}+\Delta'''_{\frac{1}{2}}}{2}$$
$$-\frac{24}{6.5.20.25}\Delta^{iv}_0+\text{etc.}$$

Again, putting $t=-\frac{1}{5}$, and changing the signs, we find for the first difference next preceding a given function

$$u_0-u_{-\frac{1}{5}}=\frac{1}{5}\frac{\Delta'_{-\frac{1}{2}}+\Delta'_{\frac{1}{2}}}{2}-\frac{1}{50}\Delta''_0-\frac{24}{6.5.25}\frac{\Delta'''_{-\frac{1}{2}}+\Delta'''_{\frac{1}{2}}}{2}$$
$$+\frac{24}{6.5.20.25}\Delta^{iv}_0-\text{etc.}$$

The difference of these quantities gives the required second difference, which we find to be

$$\delta''_0=\tfrac{1}{25}\Delta''_0-\tfrac{2}{625}\Delta^{iv}_0=\tfrac{1}{25}(\Delta''_0-\tfrac{2}{25}\Delta^{iv}_0).\;\ldots\ldots\ldots\ldots(19)$$

As an example and exercise we show the interpolation of logarithms when every fifth logarithm is given:

Number.	Logarithm.	δ'	δ''	Δ'	Δ''
1000	3·000 000 0				
				+21 661	
1005	**3·002 166 1**		**−4·32**		−108
		4319·2			
1006	·002 598 0		−4·31		
		4314·9			
1007	·003 029 5		−4·30		
		4310·6			
1008	·003 460 6		−4·30	+21 553	
		4306·3			
1009	·003 891 2		−4·29		
		4302·0			
1010	**3·004 321 4**		**−4·28**		−107
		4297·7			
1011	·004 751 2		−4·27		
		4293·5			
1012	·005 180 5		−4·26		
		4289·2			
1013	·005 609 4		−4·23	+21 446	
		4285·0			
1014	·006 037 9		−4·20		
		4280·8			
1015	**3·006 466 0**		**−4·16**		−104
				+21 342	
1020	3·008 600 2				

19. Numerical differentiation and integration.

When the numerical values of a function for a series of equidistant values of its argument are given, both the differential coefficients of the function, and its integral between any two values of the argument, may be found.

From the successive differences of the values of a quantity we may find not only intermediate values, but the derivatives as to the argument. Taking as a unit of the argument t the intervals of the series, we find, by expressing (9) in powers of t and differentiating,

$$\left.\begin{aligned} u_t = u_0 + t(\Delta_0' - \tfrac{1}{6}\Delta_0''' + \tfrac{1}{30}\Delta_0^{\mathrm{v}} - \ldots) &+ \tfrac{t^2}{2}(\Delta_0'' - \tfrac{1}{12}\Delta_0^{\mathrm{iv}} + \ldots) \\ &+ \tfrac{t^3}{3}(\Delta_0''' + \tfrac{1}{4}\Delta_0^{\mathrm{v}} + \ldots) \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots \\ D_t u_t = \Delta_0' - \tfrac{1}{6}\Delta_0''' + \tfrac{1}{30}\Delta_0^{\mathrm{v}} - \ldots + t(\Delta_0'' - \ldots) + \ldots & \\ D_t^2 u_t = \Delta_0'' - \tfrac{1}{12}\Delta_0^{\mathrm{iv}} + \ldots & \\ \text{etc.} \qquad\qquad \text{etc.} & \end{aligned}\right\} \quad (20)$$

If the interval of the argument is k units the nth derivative thus obtained will be k^n times too large, and we shall have for the true values

$$\left.\begin{aligned} \frac{du_t}{dt} &= \frac{1}{k}(\Delta_0' - \tfrac{1}{6}\Delta_0''' + \tfrac{1}{30}\Delta_0^{\mathrm{v}} + \text{etc.}) \\ \frac{d^2u_t}{dt^2} &= \frac{1}{k^2}(\Delta_0'' - \tfrac{1}{12}\Delta_0^{\mathrm{iv}} + \text{etc.}) \end{aligned}\right\} \quad \ldots\ldots\ldots (21)$$

20. Development in powers of the time.

The preceding formulae enable us to develop a quantity in powers of the time when we have given a sufficient number of special values of the quantity for equidistant epochs. As an example of the method we shall take the values of a certain quantity λ, which enters into the theory of precession, and for which we shall hereafter derive the values shown in the following table. The table shows also the successive orders of differences, so far as they are required.

Epoch.	λ	Δ′	Δ″	Δ‴	
1850	0″·000				
		6″·114			
1900	6 ·114		−1″·193		
		4 ·921		−1	
1950	11 ·035		−1 ·192		(22)
		3 ·729		0	
2000	14 ·764		−1 ·192		
		2 ·537		0	
2050	17 ·301		−1 ·192		
		1 ·345			
2100	18 ·646				

In this problem it is best to first take a date near the middle of the series as the initial one. We shall therefore count t from 1950, using 50 years as the unit. We then have

$$\Delta_0' = \tfrac{1}{2}(4''\cdot921 + 3\cdot729) = 4''\cdot325,$$
$$\Delta_0'' = -1''\cdot192,$$
$$\Delta_0''' = -0''\cdot0005.$$

By substituting these values in Stirling's formula, the development becomes

$$\lambda = 11''\cdot035 + 4''\cdot325t - 0''\cdot596t^2. \qquad\qquad (23)$$

For further use we transfer the epoch from 1950 to 1850, and express the time in terms of the century as the unit. Putting t for the time thus expressed, we have

$$t = 2(T-1),$$
$$t^2 = 4(T^2 - 2T + 1).$$

Substituting these values of t and t^2, we find

$$\lambda = 0''\cdot001 + 13''\cdot418T - 2''\cdot384T^2. \qquad\qquad (24)$$

In applying this process it must be noted that, if we make the development conform exactly to the special numerical values, the imperfections of the decimals may result in adding fictitious terms to the series. We must therefore drop all coefficients of powers of t from the point where the differences cease to be regular. In the present case it is evident that the second differences may be taken as constant; we therefore stop at the term in t^2.

After obtaining the development in this way it is advisable to compare its results with the special values of the quantity, and correct the coefficients so as to secure the best representation of the given quantities. In the present case we shall find that the six special values of λ are all represented exactly by the expression (24) except the first. The rigorous value is 0 at the epoch 1850. If we drop the first term from (24), all the other values will be in error by $0''\cdot001$. We can lessen this difference by a slight change in the coefficients of T and T^2, adding

$$\Delta\lambda = 0''\cdot0010T - 0''\cdot0002T^2.$$

Applying this correction to (24), the definitive value of λ will then be

$$\lambda = 13''{\cdot}4190T - 2''{\cdot}3842T^2.$$

The special values for the six epochs, and their deviations from the computed values (22), are as follows:

1850	0″·0000	Dev. = 0
1900	6 ·1134	−6
1950	11 ·0348	−2
2000	14 ·7640	0
2050	17 ·3012	+2
2100	18 ·6462	+2

The integral of the function u may be found by integrating its value developed in powers of the time. But this method is limited in its application. The integrals through long periods of time are found by the process of mechanical integration, which, not being necessary in spherical astronomy, is not developed in the present work.

NOTES AND REFERENCES.

The mathematical theory of interpolation is developed by Gauss in a memoir on that subject; but from a point of view wholly different from that of the practical computer.

Oppolzer in his *Lehrbuch zur Bahnbestimmung*, vol. ii., Introductory Chapter, develops the general formulae with great fulness, having especially in view the process of mechanical integration.

Rice, Herbert L., *The Theory and Practice of Interpolation*, Lynn, Mass., Nichols Bros., 1899, is a very copious and extended exposition of the various applications of the method, which can be recommended to the computer who has much of this work to do.

CHAPTER III.

THE METHOD OF LEAST SQUARES.

Section I. Mean Values of Quantities.

21. The "method of least squares" is a subject which requires a volume for its full treatment. But the most essential principles involved in it, and the simplest of the processes which are applied in much every-day astronomical work, can be set forth in a smaller compass.

The method has its origin in the fact that when we aim at the highest precision in astronomical measurement, we find the results of our measures to be affected by small errors due to a multiplicity of unavoidable causes. Some of these are in the nature of accidents; of others the causes are known in a general way, but cannot be obviated or determined in detail. The result is that a perfect agreement between two observations is never to be expected. The combination of discordant measures so as to derive the most likely result thus becomes an important part of the astronomer's work.

22. Distinction of systematic and fortuitous errors. The errors in astronomical measurement are divided into two classes, one called *systematic*, the other *accidental* or *fortuitous*, according to the nature of their causes.

Systematic errors are those arising from causes which continue their action through a series of observations, or are in any way governed by a determinable law. Examples of such causes are: Changes of temperature which may cause an instrument to give a different result in summer and in winter, or during the day or in the night; varying conditions of the atmosphere, resulting in its refracting light differently on one night from what it does on another; habits of the observer leading him to make an error of the same general nature in a whole series of observations; imperfections in the construction of an instrument leading to its results being always erroneous in a more or less regular way.

A systematic error of which the amount is always the same is called *constant*. This term is also applied to the mean value of any systematic error in a series of observations. An example of a constant error is offered by a scale of millimetres or angles being too long or too short. It is evident that, in every such case, all the measures made with the scale will be too small or too large by a corresponding amount. If the scale is correct at a certain standard temperature, and the observer uses it at another temperature, always higher or lower than the standard one, the general mean of the systematic errors will be those corresponding to the mean of the actual temperatures.

The general, though not the universal, rule is that systematic errors admit of investigation and determination, so that we may with more or less certainty determine the proper corrections to be applied in order to annul their effect.

Accidental or *fortuitous* errors are those of which the causes are so variable and transient that the resulting errors elude investigation. For example, if an observer seeks to bisect the segment of a line by his eye and by estimation, there must be a range of accidental error, at least equal to the smallest space perceptible to the senses. The undulations of the air, which never entirely cease, cause the image of a star, as seen in a telescope, to be continually affected by a small and irregular motion, or change of form. An error which cannot be estimated in advance will therefore be made by the observer when he attempts to bisect the image with a spider line.

The general rule, in astronomy at least, is that such accidental errors are the result of the separate action of a multiplicity of causes, too variable and complex to be individually determined or even defined. The theory of such errors and the reduction of their injurious effect to a minimum form the basis of the subject commonly known as the Method of Least Squares.

This method properly applies only to accidental errors. The best results we can derive by it will still be affected by the constant or average effect of all the systematic errors to which the observations are liable. These we must determine as best we can in each case, and regard the uncertainties of the determination as belonging to the class of accidental sources of error.

23. The arithmetical mean and the sum of the squares of residuals. We begin with the simplest case, which is that of the repeated measurement of a constant quantity, which we may call x.

Let the individual results of n measurements be that this quantity is found to have the values

$$x_1, x_2, x_3, \dots x_n.$$

We may express these results by saying that they lead to the discordant equations

$$\begin{aligned} x &= x_1, \\ x &= x_2, \\ \vdots &\quad \vdots \\ x &= x_n; \end{aligned}$$

or

$$\left.\begin{aligned} x - x_1 &= 0 \\ x - x_2 &= 0 \\ x - x_3 &= 0 \\ \vdots \quad \vdots &\quad \vdots \\ x - x_n &= 0 \end{aligned}\right\} \quad \dots\dots\dots\dots (1)$$

and the question is how, from all these equations, we are to conclude upon the best value to adopt for x.

Let us first regard x as an indeterminate quantity, to which we may, as an hypothesis, assign any value at pleasure. We may form the deviations of any such value from the observed values. Putting

$$r_1, r_2, r_3, \dots r_n$$

for these deviations, we have

$$\left.\begin{aligned} x - x_1 &= r_1 \\ x - x_2 &= r_2 \\ \vdots \quad \vdots &\quad \vdots \\ x - x_n &= r_n \end{aligned}\right\} \quad \dots\dots\dots\dots (2)$$

Now let us form the sum of the squares of these residuals, which we call Ω.

$$\Omega = r_1^2 + r_2^2 + \ldots + r_n^2$$
$$= nx^2 - 2(x_1 + x_2 + \ldots + x_n)x + x_1^2 + x_2^2 + \ldots + x_n^2.$$

This sum is a quadratic function of x; and the fundamental principle adopted is that the most likely or best value of x to be chosen is that which makes the sum Ω of the squares of the residual differences the least possible. To find this value we differentiate Ω as to x, and equate the derivative to zero. We thus find

$$\frac{d\Omega}{dx} = 2nx - 2(x_1 + x_2 + x_3 + \ldots + x_n),$$

which, being equated to zero, gives

$$x = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}. \quad \text{..................... (3)}$$

Hence, on the principle in question, the most likely value of the quantity measured is the mean result of the individual measures.

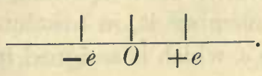
24. The probable error.

Since every astronomical result is liable to error, we need some way of expressing the amount of the liability. We may do this by assigning a quantity e such that we suppose it a certain definite chance whether the observation is in error by an amount greater or less than e. It is common to regard the chance in question as an even one; the value of e is then called the *probable error*.

The judgment that a numerical value x_1, assigned to x, is affected by the probable error e, that is to say, that the true x probably differs from x_1 by the quantity e, is expressed in the form

$$x = x_1 \pm e.$$

This means that, out of four chances, there are two that x is contained between the limits $x_1 - e$ and $x_1 + e$, and two that its true value lies without these limits. Let us lay down the value of x graphically on an axis of abscissae and measure off the value of the probable error on each side, thus



The point O here marks the adopted value of x, while $+e$ and $-e$ are laid off on each side.

Then the four chances that the true value of x lies

$$\left.\begin{array}{l}\text{To the left of } -e\\ \text{Between } -e \text{ and } O\\ \text{Between } O \text{ and } +e\\ \text{To the right of } +e\end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

are all equal.

The following problems are fundamental in the whole theory:

Problem I. *The probable error of x being $\pm e$, to find the probable error of mx*, m being a constant.

If x is contained between the limits $x+e$ and $x-e$, mx will be contained between the limits $mx-me$ and $mx+me$.

There is, therefore, the same chance that mx will be contained between these limits as there is that the true x will be between $x+e$ and $x-e$.

Hence $$\text{probable error of } mx = \pm me. \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

It may be noted that this theorem is true of values of m either greater or less than unity.

Definition. Independent quantities are such that a change in the adopted value of one will not affect our judgment of the value of the other.

Problem II. *The probable errors of several independent quantities of which the given values are q_1, q_2, … q_n being e_1, e_2, … e_n:—to find the probable error of their sum.*

Let us put

q'_i the true, but unknown, value of $q_i \ldots$, $(i=1, 2, \ldots n)$

$Q=q_1+q_2+\ldots+q_n$, the sum as given,

$Q'=q'_1+q'_2+\ldots+q'_n$, the true unknown sum.

We may then write the equations

$$\begin{aligned} q'_1-q_1&=\pm e_1,\\ q'_2-q_2&=\pm e_2,\\ &\vdots\\ q'_n-q_n&=\pm e_n,\end{aligned}$$

which will mean that, in each equation, it is an even chance whether the second member is, in absolute magnitude, greater or less than the value e, which is assigned to it.

Take the sum of all these equations,

$$Q' - Q = \pm e_1 \pm e_2 \pm e_3 \pm \ldots \pm e_n$$

and square it,

$$(Q' - Q)^2 = e_1^2 + e_2^2 + \ldots + e_n^2$$
$$\pm e_1e_2 \pm e_1e_3 \pm e_2e_3 \pm \ldots \pm e_{n-1}e_n.$$

Since the quantities e are equally likely to have positive or negative values, the sum of the terms in the last line is as likely to be positive as negative. The probable average value of this sum is therefore zero, and the most probable value of the second term of the equation becomes $e_1^2 + e_2^2 + \ldots + e_n^2$. We therefore obtain as the most likely value of $Q' - Q$,

$$E = Q' - Q = \surd(e_1^2 + e_2^2 + e_3^2 + \ldots + e_n^2), \ldots\ldots\ldots\ldots\ldots (6)$$

E being put for the probable error of the sum Q. We therefore have the theorem:

The probable error of the sum of any number of independent quantities is equal to the square root of the sum of the squares of the probable errors of the individual quantities.

We note that the seeming difference between the conclusions in the two cases of Problems I. and II. arises from the premise of the second case that the quantities are independent.

Problem III. *To find the probable error of a linear function of several independent quantities in terms of the probable errors of the separate quantities.*

Let the quantities with their probable errors be

$$x_1 \pm e_1, \quad x_2 \pm e_2, \quad x_3 \pm e_3, \quad \ldots,$$

and let the linear function be

$$X = ax_1 + bx_2 + cx_3 + \ldots.$$

By Problem I. we have

Probable error of $ax_1 = \pm ae_1$,

" " $bx_2 = \pm be_2$,

" " $cx_3 = \pm ce_3$,

..................................,

and then, by Problem II.,

$$\text{Probable error of } X = \surd(a^2e_1^2 + b^2e_2^2 + c^2e_3^2 + \ldots), \ldots\ldots\ldots (7)$$

which is the required result.

PROBLEM IV. *To find the probable error of the arithmetical mean of several independent quantities each having the same probable error.*

The arithmetical mean of m quantities $x_1, x_2 \dots x_m$ being the linear function

$$\frac{x_1}{m}+\frac{x_2}{m}+\frac{x_3}{m}+\dots+\frac{x_m}{m},$$

this case becomes identical with that of Problem III. by putting

$$a=b=c=\dots=\frac{1}{m},$$

$$e_1=e_2=e_3=\dots=e.$$

Hence, by substitution in (7),

$$\text{Probable error}=\sqrt{\frac{me^2}{m^2}}=\frac{e}{\sqrt{m}}. \qquad \text{(8)}$$

Hence: *The probable error of the mean result of several observations of the same quantity is inversely as the square root of their number.*

25. Weighted means.

The arithmetical mean of several results can be the most likely value only in the case when we have no reason to prefer any one of the results to any other. But if some of the results have a smaller probable error than others, it is evident that we should esteem them of greater reliability in reaching a conclusion.

To find how we must modify the principle of least squares in this case, suppose that the given results

$$x_1,\ x_2, \dots x_n,$$

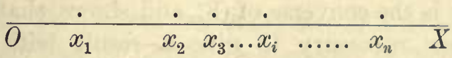
instead of being individual results of separate observations, are each the mean of several observations of one and the same quantity throughout, namely:

x_1, the mean result of m_1 obs.;
x_2, " " " " m_2 "
⋮ " " " " ⋮ "
x_n, " " " " m_n "

The mean we should adopt is that of the original results. To form their sum we note that, since x_1 is the mean of m_1 quantities, the sum of these quantities must have been m_1x_1. Hence the sum of all the products m_1x_1, m_2x_2, etc., will be the sum of the original results, while the number of the latter is the sum of the m's. Hence the required mean is

$$x = \frac{m_1x_1 + m_2x_2 + \dots + m_nx_n}{m_1 + m_2 + \dots + m_n}. \qquad \text{(9)}$$

In this expression the factors m are termed *weights* and the result is called a *weighted mean*, or a *mean by weights* of the quantities $x_1 \dots x_n$



This conception of a weighted mean has a mechanical analogue. If we lay off on an axis OX the values of x_1, x_2, etc., the arithmetical mean of all the measures from O to X corresponds to the distance from O to the centre of gravity of the points $x_1, x_2, \dots,$ when all are assigned equal weights. If we imagine the points to have different weights, the position of the centre of gravity is that of the weighted mean of the quantities.

It is evident that if all the numbers m are multiplied by any common factor, the resulting value of x will not thereby be changed. Hence we may take for the weights any system of numbers proportional to the respective numbers of observations from which each separate result is derived. In other words, we may find the weights by multiplying or dividing the numbers of observations by any common factor or divisor. We represent weights, in a general way, by the symbols

$$w_1, \ w_2, \dots w_n.$$

26. Relation of probable errors to weights.

In (9) the weights m are the respective numbers of observations on which each x depends. But, suppose that, instead of the number of observations being given, we have given the probable error of each measure or series of measures. It is evident that the final result to be derived should depend on these probable errors, irrespective of the number of observations which enter into each result.

It has been shown (Prob. IV.) that the probable error of any number of observations, each having the same individual probable error, is inversely proportional to the square root of the number. If e_0 be the probable error of a single observation, that of the mean of m observations will, therefore, be

$$e = \frac{e_0}{\sqrt{m}}.$$

Hence $$m = \frac{e_0^2}{e^2}. \qquad \text{.............................(10)}$$

This result is the converse of (8), and shows that the number of observations necessary to give a result with an assigned probable error is inversely as the square of that error.

It follows that if we have given the results

$$x_1,\ x_2, \ldots x_n,$$

with the respective probable errors

$$e_1,\ e_2, \ldots e_n,$$

we may choose at pleasure a quantity e_0 as the probable error of a fictitious standard observation, to which we assign the weight 1, and then find the series of numbers

$$w_1 = \frac{e_0^2}{e_1^2},\quad w_2 = \frac{e_0^2}{e_2^2}, \ldots,\quad w_n = \frac{e_0^2}{e_n^2}, \qquad \text{................(11)}$$

which will be the respective numbers of fictitious observations the means of whose results will have the given probable errors. There is no need that the numbers w shall be integers; nor is there commonly any practical advantage in writing their values with more than a single significant digit, or, at most, a pair of digits when the first digit is 1.

As a concrete example, suppose the seconds of mean declination of a star at a certain epoch, as determined at several observatories, with their probable errors, to be

$$\delta = 3''{\cdot}1;\ 3''{\cdot}7;\ 2''{\cdot}9,\ 3''{\cdot}2,\ 3''{\cdot}7,$$

$$e = \pm 0''{\cdot}22;\ 0''{\cdot}25;\ 0''{\cdot}18;\ 0''{\cdot}13;\ 0''{\cdot}40.$$

As a convenient round number we choose $e_0 = 0''\cdot50$ as the probable error of a fictitious standard observation. We then have the following computation from (11) and (9):

i.	e_i^2.	w_i.	$w_i\delta$.
1	0″·048	5	15″·5
2	0 ·062	4	14 ·8
3	0 ·032	8	23 ·2
4	0 ·017	15	48 ·0
5	0 ·160	2	7 ·4
		34	108″·9

Weighted mean: $108''\cdot9 \div 34 = 3''\cdot20$.

Problem V. *To find the probable error of a weighted mean.*

Since the weighted mean may be regarded as that of a certain number of standard observations, its probable error is given by (8). The number in question being

$$w_1 + w_2 + \ldots + w_n \equiv W, \qquad (12)$$

the probable error e of x is

$$e = \frac{e_0}{\sqrt{W}}, \qquad (13)$$

or the quotient from dividing the probable error of a fictitious observation of weight 1 by the square root of the sum of the weights.

The same result may be reached in an elegant way by Problem III. A weighted mean is a linear function of the quantities whose mean is taken, of which the coefficients are

$$a = \frac{w_1}{W}, \quad b = \frac{w_2}{W}, \text{ etc.}$$

Substituting in (7) these values of a, b, etc., and putting for e_1^2, e_2^2, etc., their values from (11)

$$e_i^2 = \frac{e_0^2}{w_i},$$

the probable error reduces to the expression (13).

27. Modification of the principle of least squares when the weights are different.

It is evident that, when we take a weighted mean of several quantities, the sum of the squares of the residuals can no longer be a minimum, because this is the case with the unweighted mean. To find the corresponding function which is a minimum, let the quantities whose weighted mean is taken be, as before,

$$x_1, x_2, x_3, \ldots x_n,$$

and their probable errors

$$e_1, e_2, e_3, \ldots e_n.$$

Their respective weights will then be

$$w_1 = \frac{e_0^2}{e_1^2}, \quad w_2 = \frac{e_0^2}{e_2^2} \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

If we multiply the equations (1) or (2) by the respective factors

$$\sqrt{w_1} = \frac{e_0}{e_1}, \quad \sqrt{w_2} = \frac{e_0}{e_2}, \text{ etc. } \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

it follows from (5) that the probable errors of the products

$$\sqrt{w_1}x_1; \quad \sqrt{w_2}x_2; \ldots \quad \sqrt{w_n}x_n$$

will all be equal to e_0. By applying this multiplication to equation (2) the second members will become $\sqrt{w_1}r_1, \sqrt{w_2}r_2, \ldots,$ and it is the sum of the squares of these products, that is, the function

$$w_1r_1^2 + w_2r_2^2 + \ldots + w_nr_n = \Omega, \ldots\ldots\ldots\ldots\ldots\ldots(16)$$

which should be a minimum. If we substitute for $r_1, r_2, \ldots$ their values (2), differentiate as to x, and equate to 0, we shall have

$$w_1(x - x_1) + w_2(x - x_2) + \ldots = 0,$$

which will give for x the weighted mean of x_1, x_2, etc.

28. Adjustment of quantities.

It sometimes happens that we know in advance some relation which a system of measured quantities should satisfy. For example, if at a point O on a horizontal plane, surrounded by points $A, B, C, D, \ldots K$, lying on the plane in various directions, we measure the consecutive angles

$$AOB, BOC, COD, \ldots KOA,$$

the last angle carrying us round to the starting direction, then we know that the sum of these angles should be 360°. If the measures were free from error, then, on adding them together, the sum would be exactly 360°. If the actual sum is different from this by a quantity Δ, we know that Δ is the algebraic sum of all the errors of the measures. The problem then is to find the most likely system of corrections, which, being applied to the individual measures, will reduce their sum to 360°. In doing this we are *adjusting* the measures so as to fulfil the required conditions, for which reason the process is called *adjustment.*

If the measures are all of one weight, the process of adjustment is this:—Putting

$$a_1, a_2, a_3, \dots a_n,$$

the measured values of the successive angles, then

$$\Delta = a_1 + a_2 + a_3 + \dots + a_n - 360°$$

will be the sum of the errors. The most likely adjustment will then be to divide the errors equally among all the angles. If we put

$$\delta = \frac{\Delta}{n},$$

the concluded values of the separate angles will be

$$a_1 - \delta, \ a_2 - \delta, \dots \ a_n - \delta,$$

the sum of which fulfils the required condition of being 360°.

If the weights are unequal, let us put for the weights of the n measures

$$a_1, a_2, a_3, \dots a_n$$

the symbols

$$w_1, w_2, w_3, \dots w_n,$$

and let the respective corrections be

$$h_1, h_2, h_3, \dots h_n.$$

The sum of these quantities must satisfy the condition

$$h_1 + h_2 + h_3 + \dots + h_n = -\Delta, \quad \dots\dots\dots\dots\dots\dots(17)$$

while, in accordance with the general principle, the function

$$\Omega = w_1 h_1^2 + w_2 h_2^2 + \dots + w_n h_n^2$$

must be the least possible. Hence the equation

$$w_1 h_1 dh_1 + w_2 h_2 dh_2 + \dots + w_n h_n dh_n = 0 \quad \dots\dots\dots\dots(18)$$

must be satisfied for all values of the dh's which satisfy the equation given by the differentiation of (17), namely

$$dh_1 + dh_2 + dh_3 + \dots + dh_n = 0. \quad \dots\dots\dots\dots\dots(19)$$

These conditions must hold true for every admissible infinitesimal change in the values of the h's. To find the values which satisfy the conditions, we multiply (19) by an indeterminate factor λ and subtract the product from (18), thus obtaining

$$(w_1h_1-\lambda)dh_1+(w_2h_2-\lambda)dh_2+\ldots+(w_nh_n-\lambda)dh_n=0.$$

In order that this equation may be satisfied for all values dh_1 dh_2, etc., we must have

$$w_1h_1-\lambda=0,\ w_2h_2-\lambda=0, \text{ etc.,}$$

or

$$w_1h_1=w_2h_2=w_3h_3\ldots=w_nh_n=\lambda.$$

Hence

$$\left.\begin{array}{l} h_1=\dfrac{\lambda}{w_1} \\ h_2=\dfrac{\lambda}{w_2} \\ \vdots \quad \vdots \\ h_n=\dfrac{\lambda}{w_n} \end{array}\right\}. \qquad \ldots\ldots\ldots\ldots(20)$$

Equating the sum of these quantities to $-\Delta$ gives us an equation for determining λ;

$$\lambda=\frac{-\Delta}{\dfrac{1}{w_1}+\dfrac{1}{w_2}+\ldots+\dfrac{1}{w_n}}, \qquad \ldots\ldots\ldots\ldots(21)$$

which substituted in (20) gives the values of the adjusted corrections h.

It is interesting and instructive to note the form of this result when one of the measures has the weight 0, or, in other words, is regarded as entirely worthless. Let this measure be the first so that $w_1=0$.

Then (21) will give $\lambda=0$

and (20) will give $h_2=h_3=\ldots=h_n=0,$

but will leave h_1 indeterminate. But its value is found by using, instead of (20) the expression (17). This gives at once

$$h_1=-\Delta.$$

In other words we are obliged, in this special case, to determine the faulty h from the sum of all the remaining measures, a conclusion evident in advance.

Section II. Determination of Probable Errors.

29. Of probable and mean errors.

The preceding theory assumes as one of the data to be given a certain quantity called a *probable* error. The definitions of this and certain associated quantities, and the methods of determining them, are now to be considered. The following is the logical basis of the subject:

(1) In a rigorous sense, an *error* consists in the deviation of an observed value from an absolutely true value. But the latter quantity is never considered as actually known. Hence, what we have to take as an error is the deviation of an individual value from the best value that we are able to determine. In certain cases the term *residual* or *residual difference* is applied to this quantity. But, with the limitations we have expressed, the use of the usual term "error" should cause no misconstruction.

(2) The probable error e, as we have defined it, is determined by the condition that there is an equal chance of an error, in any one case, being greater or less than e in absolute value.

But the reasoning, as it has been set forth, is equally valid for the case when, instead of taking e for the amount of that error which there is an even chance of committing, we take the value of an error having a different probability from this. We may, for example, take an error of which there are three chances to one against committing. In this case, in the language of the theory of probabilities, the probability of an error exceeding the standard amount will be $\frac{1}{4}$. The reasoning would then remain the same throughout, the meaning of the term probable error being alone changed.

(3) We must distinguish between a *probable* and an *actual* error. The actual errors are numerical values of the residuals which we actually find in the case of any system of observation. Probable errors are errors of which there is a greater or less probability of making. Hence actual errors may be greater or less than probable ones, but, in the long run and the general average, they should correspond to each other.

(4) The term *mean error* may be used in various significations which must be distinguished. In general, the mean of several quantities is equal to the quotient of their algebraic sum by their number. If used in this sense, the mean of all the deviations of a system of quantities from their arithmetical mean would always be zero, and the term would be without significance.

Another signification of the term is the mean of the numerical values of all the errors, regardless of their algebraic sign. This is called an *arithmetical mean* error, or *average error*, but is not much used in practice.

As commonly used, the term *mean error* is that quantity whose square is the mean of the squares of the errors: that is, it is the square root of the arithmetical mean of the squares of all the errors.

It is easily shown that this mean is always greater than the arithmetical mean of the errors, except in the case when all the errors are equal.

For, let us take

$$\left.\begin{array}{l}\epsilon, \text{ the mean}\\ v, \text{ the average}\end{array}\right\}, \text{ as just defined,}$$

of the n errors

$$e_1,\ e_2,\ \ldots\ e_n.$$

Let us also take the difference between each individual error and the average, and call it c, so that we have

$$\begin{aligned} e_1 &= v \pm c_1,\\ e_2 &= v \pm c_2,\\ \vdots\ &\quad \vdots\quad \vdots\\ e_n &= v \pm c_n. \end{aligned}$$

Now, take the sum of the squares of these equations:

$$\Sigma e^2 = nv^2 + \Sigma c^2 + 2v\Sigma c.$$

The first member is, by definition, $n\epsilon^2$, and the last term vanishes, because $\Sigma c = 0$. Hence

$$\epsilon^2 = v^2 + \frac{\Sigma c^2}{n}.$$

The excess of ϵ^2 over v^2 is a positive quantity, which vanishes only in the special case when the e's are all equal (making the ϵ's all zero).

Here again we must distinguish between a *probable* mean error and the *actual* mean error in any given case. The latter is a numerical result actually found from the residuals; the former is defined as the mean error which would be found in making an infinite number of observations of the same kind, and, so far as can be determined, under the same conditions, as those actually made.

The actual mean is found only from the special observations in question, but, in determining a probable mean, we may take into consideration all the data at our disposal for its determination.

30. Statistical distribution of errors in magnitude.

The method of dealing with fortuitous errors rests upon the law of their statistical distribution in magnitude, that is to say, the respective probabilities of making errors of different magnitudes.

The following are the assumed general laws of distribution, from which, however, there may be deviations in special or extraordinary cases. To the latter the theory of the subject does not apply. In the cases to which it does apply, the principles are:

(1) *Positive and negative errors of any given magnitude are equally probable.*

It is readily seen that there are many kinds of investigation in which this law does not hold true. For example, in weighing, impurities in the substance weighed will always result in making the apparent weight greater than that of the pure substance. In astronomy, however, the law is very near the truth.

(2) *In any class of observations, the probability of an error continually diminishes with its magnitude, and we can always set a limit beyond which the probability of an error shall be as small as we please.*

It is, however, impossible to set a limit which an error may reach, but can never exceed. We can only say that, the larger a possible error, the more unlikely it should be to occur.

The preceding laws are commonly embodied in the following formula:—If we put h for a certain modulus of error, then the infinitesimal probability that the error shall lie between the limits x and $x+dx$ is assumed to be given by the formula:

$$dp=\frac{1}{h\sqrt{\pi}}e^{-\frac{x^2}{h^2}}dx. \quad \text{.........................(22)}$$

This formula may be graphically represented by the curve shewn in Fig. 2, in which, if the abscissa of any point represents the magnitude of an error, the ordinate at that point is proportional to the probability of the error. The point P marks the probable error and M the mean error.

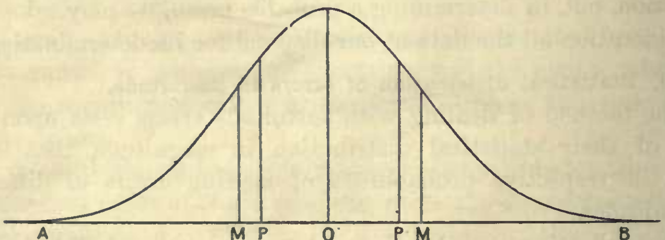


FIG. 2.

If we have an indefinitely great number of errors distributed in magnitude according to this law, we may represent each error by measuring off its magnitude from the origin O to the right or left, according as the error is positive or negative, and then marking it by a point. Since the ordinate at each point is proportional to the number of errors of corresponding magnitude, it follows that if we scatter the points along the ordinate they will be equally distributed over the area contained between the curve and the axis of abscissas AB. The total number of errors between any two limits will be proportional to the area contained between the corresponding ordinates.

The probabilities that an error will exceed certain amounts are

The probable error itself -	0·500
The mean error - -	0·318
2 times the probable error	0·177
3 " " " "	0·043
4 " " " "	0·007

Thus the area PP will be 1/2 and that between the ordinates of mean error MM will be 0·318 of the entire area of the curve. At the points A and B, corresponding to errors of about four times the probable error, the curve approaches so near the axis that only seven errors out of one thousand should reach these limits.

The preceding law of error is considered the normal law, and on it the theory of the subject is commonly based. But, although it is a law to which the errors will commonly approximate when the observations are carefully made, it cannot be regarded as practically universal. Indeed, in practice, the general rule is that large errors are more common than the normal law would lead us to infer. For example, an error five times as great as the probable one should, on the theory, occur only once in 1300 times, but practically, it will be found to occur much oftener. The theory is, however, adopted because of the simplicity and elegance of the methods based upon it.

The practical astronomer has also to recognize the occasional, and perhaps the frequent occurrence of errors which seem abnormally large. Such an error may be of a magnitude so great that no question can arise as to its retention or rejection. A wrong figure may have been written down, or a wrong graduation read by the observer. But when the magnitude of the error is such that it cannot be regarded as morally impossible, the question of dealing with it becomes one of great difficulty, to be settled by common sense and sound judgment rather than by any theory. The general rule is that, if the magnitude of a residual exceeds the value which we could reasonably suppose a fortuitous error to have among a number of observations no greater than that which we are combining, we must regard it as abnormal, and reject the result affected by it.

31. Method of determining mean or probable errors.

In combining observations an important problem is that of inferring the probable error to which any one observation should be regarded as liable. This may be done in two ways:

(1) We may know from experience that observations of a certain class, made at a certain observatory, or by a certain

observer, are affected by a probable error of a certain amount. For example, meridian observations of declination are affected by a probable error which may lie between $\pm 0''\cdot 2$ and $\pm 0''\cdot 5$. When made on objects near the horizon, the p.e. may even exceed the latter limit. That of good observations in R.A. commonly lies between $\pm 0^{s}\cdot 018$ and $\pm 0^{s}\cdot 035$.

(2) The probable error of the individual observations of a series may be determined by their mutual discordances, or the deviation of each from the mean of all. A result thus reached will be more reliable the greater the number of observations. In developing the method of doing this we begin with a numerical example, illustrating the combination of observations, and the determination and treatment of residuals.

Twelve observations of the north polar distance of Aldebaran, made at the Royal Observatory, Greenwich, during the year 1899, gave the results shown in the first column of the following table, when reduced to the beginning of the year. The degrees and minutes, 73° 41′, are omitted, being the same for the whole series.

COMBINATION OF GREENWICH OBSERVATIONS OF ALDEBARAN, 1899.

	Sec. of N.P.D.	res.	r^2.
Feb. 9	36″·55	−0″·49	0″·24
Mar. 23	38 ·04	+1 ·00	1 ·00
24	37 ·63	+0 ·59	0 ·35
Apr. 10	38 ·17	+1 ·13	1 ·28
19	36 ·87	−0 ·17	0 ·03
May 6	36 ·29	−0 ·75	0 ·56
June 16	36 ·64	−0 ·40	0 ·16
July 10	36 ·45	−0 ·59	0 ·35
Nov. 16	37 ·74	+0 ·70	0 ·49
Dec. 11	36 ·11	−0 ·93	0 ·86
13	37 ·18	+0 ·14	0 ·02
14	36 ·83	−0 ·21	0 ·04
Sum, -	84″·50	+0″·02	5″·38
Mean, -	37 ·04		

The units and decimals of seconds being added, give the sum 84″·50, and the mean 7″·04. There is no use in carrying the division beyond the second decimal, which might, indeed, have been omitted from all the separate observations without detracting from the precision of the result.

This mean being subtracted from each separate result gives the apparent residual errors of the latter found in the third column. In mathematical theory their algebraic sum should be 0. As a control upon the accuracy of the mean, we form the sum, and find it to be +0″·02. This is because the remainder 0″·02 has been neglected in dividing to form the mean.

We next take the square of each residual, dropping unnecessary decimals, and find the mean value of all the squares. But, in forming this mean, we use a divisor less by 1 than the number of observations, for a reason now to be shewn.

In determining a probable error we must, in effect at least, express the result as a linear function of the observed quantities. So we express the residuals as linear functions of the observed results. Let the latter, n in number, be

$$x_1,\ x_2,\ x_3, \ldots x_n,$$

the mean is the sum of these divided by n. Subtracting this from any x, say x_1, we find the residual to be

$$r_1 = \left(1 - \frac{1}{n}\right)x_1 - \frac{1}{n}x_2 - \frac{1}{n}x_3 - \ldots - \frac{1}{n}x_n. \qquad (23)$$

We now put ϵ for the unknown mean error of each x. Then, by § 24, Eq. (7), the square of the probable mean error of the linear function (23) is

$$\left\{\left(1 - \frac{1}{n}\right)^2 + \frac{n-1}{n^2}\right\}\epsilon^2 = \frac{n-1}{n}\epsilon^2. \qquad (24)$$

This quantity is an expression for the probable value of the square of any one residual, taken at random. We have n such squares which, when equated to the respective residuals, give us n probable equations of the form

$$\frac{n-1}{n}\epsilon^2 = r_i^2 \ (i = 1, 2, \ldots n).$$

The sum of these n equations gives for ϵ the probable value

$$\epsilon^2 = \frac{\Sigma r^2}{n-1}.$$

In the example before us, we have $n = 12$. Hence

$$\epsilon^2 = \frac{\Sigma r^2}{11} = \frac{5\cdot 38}{11} = 0\cdot 49$$

and $$\epsilon = \pm 0''\cdot 70.$$

This is the probable value of the *mean error* already defined, which the writer deems the best to use, as the expression of the uncertainty of a result. But it is quite common to use the so-called probable error, or the error which there seems to be an even chance of exceeding in the case of any one observation. Assuming the respective probabilities of errors of different magnitudes to follow the normal law stated in § 30, it can be shown that the probable error is equal to the mean error multiplied by the factor

$$\sigma = 0\cdot 6745.$$

In the present case, this gives $\pm 0''\cdot 47$ as the probable error of a single observation.

Finally, the probable error of the result is found by dividing the probable error of one observation by the square root of the number of observations. Thus we may express the mean result of the observations given above, together with its uncertainty, in the form

N.P.D. of Aldebaran $= 73° \; 41' \; 37''\cdot 04 \pm 0''\cdot 20$ (m.e.) or $\pm 0''\cdot 14$ (p.e.).

32. Case of unequal weights.

Let us now consider the more complex case in which the results to be combined are of different weights. As a numerical example, we take the following six measures of the interval of time taken by light in passing from Fort Myer to the Washington Monument and back, made by the author in 1882. The intervals are expressed in millionths of a second.

The weights are assigned according to the number of turns of the revolving mirror from which the ray was reflected, and all other circumstances affecting the quality of the result.

In taking a mean by weights there is no need of multiplying the whole of any one result by its weight. We may divide each x into two parts, the one an arbitrary quantity x_0, the same for all, the other the difference between x and x_0, say Δ. Then, we take the mean of all the Δ's, and add it to x_0. In the computation we have taken $x_0 = 24{\cdot}82$, and multiplied the excess Δ of the result over x_0, by the weight.

The products, $w\Delta$, are found in the fourth column, and divided by $\Sigma w = 30$ to form the weighted mean. In forming the residuals, we transfer the decimal point to follow the thousandth of millionths place.

1882.	Interval of time.	Weight.	$w\Delta$.	r.	wr.	wr^2.
July 24	24·828	4	32	+0·4	+ 1·6	1
26	24·828	3	24	+0·4	+ 1·2	0
Aug. 9	24·822	2	4	−5·6	−11·2	63
10	24·825	5	25	−2·6	−13·0	34
11	24·828	6	48	+0·4	+ 2·4	1
29	24·831	6	66	+3·4	+20·4	69
30	24·827	4	28	−0·6	− 2·4	1
Sum, -	- —	30	227	—	− 1·0	169
Mean,	- 24·8276	—	—	—	—	28·2

Each residual is then multiplied by the weight, and the algebraic sum of the products, which should vanish, taken as a control. The sum −1·0 is the remainder neglected in dividing by 30, the sum of the weights.

We next multiply each wr by r, so as to form wr^2. In doing this, there is never any use in carrying the product beyond two significant figures in the majority of the results, so we drop the decimals, and by adding find

$$\Sigma wr^2 = 169.$$

From this the probable mean error is derived by the following investigation:

33. To find the probable mean error when the weights are unequal.

Let W be the sum of the weights $w_1, w_2, \ldots w_n$, and let r_i be any residual, $x_i - x$. Expressing the latter as a linear function of the observed quantities by subtracting the weighted mean of the x's from any one x, say x_i, we have

$$r_i=\left(1-\frac{w_i}{W}\right)x_i-\frac{w_1}{W}x_1-\frac{w_2}{W}x_2-\ \dots\ -\frac{w_n}{W}x_n, \quad\dots\dots\dots(25)$$

the ith term being omitted in the last set of terms, because already included in the first term of the set. We put ϵ for the mean error corresponding to weight 1. We shall then have, for the square of the mean error of any one of the x's, say x_κ by § 26, Eq. (11)

$$\frac{\epsilon^2}{w_\kappa},$$

and for the square of the probable mean error of the term $\frac{w_\kappa}{W}x_\kappa$ we shall have

$$\frac{w_\kappa^{\ 2}}{W^2}\cdot\frac{\epsilon^2}{w_\kappa}=\frac{w_\kappa\epsilon^2}{W^2}.$$

Proceeding as before, and taking for κ the successive numbers 1, 2, 3 ... n, i alone being omitted, we find the square of the probable mean error of the linear function (25) to be from § 24, Eq. (7)

$$\left(1-\frac{w_i}{W}\right)^2\frac{\epsilon^2}{w_i}+\frac{w_1\epsilon^2}{W^2}+\frac{w_2\epsilon^2}{W^2}+\dots+\frac{w_n\epsilon^2}{W^2}.$$

The sum of the weights $w_1+w_2\dots w_n$ is $W-w_i$.
This expression therefore reduces to

$$\left\{\left(1-\frac{w_i}{W}\right)^2\cdot\frac{1}{w_i}+\frac{W-w_i}{W^2}\right\}\epsilon^2=\left(\frac{1}{w_i}-\frac{1}{W}\right)\epsilon^2.$$

Reasoning as before, this is the probable value of the square of the residual r_i. Multiplying it by w_i we have

$$\left(1-\frac{w_i}{W}\right)\epsilon^2=\text{prob. } w_ir_i^2.$$

Putting $i=1, 2, \dots n$ and taking the sum of all the equations thus formed, we have the probable equation

$$(n-1)\epsilon^2=\Sigma wr^2$$

and

$$\epsilon^2=\frac{\Sigma wr^2}{n-1}. \quad\dots\dots\dots\dots\dots\dots\dots(26)$$

The square root of this expression gives the probable mean error for weight 1, which, divided by the square root of the sum of the weights, will give the probable error of the result.

In the example we have

$$n=7; \quad W=30; \quad \Sigma wr^2=169.$$

Hence
$$\epsilon^2=28{\cdot}2,$$
$$\epsilon=\pm 5{\cdot}3,$$
$$\epsilon\div\sqrt{30}=\pm 0{\cdot}97,$$

and the mean result in units of ·000 000 001 of a second is

$$\text{Time}=24\,827{\cdot}6\pm 0{\cdot}97 \text{ (m.e.) or } \pm 0{\cdot}66 \text{ (p.e.)}.$$

The probable error is therefore less than the millionth part of the thousandth of a second, so far as it can be inferred from the discordance of the results.

Section III. Equations of Condition.

34. Elements and variables.

Many problems of astronomy are of the following character: We have certain varying quantities which we may call

$$x, \; y, \; z, \text{ etc.},$$

of which we may determine the values at certain moments by direct observation. These quantities are known functions of the time t, and of other quantities

$$a, \; b, \; c, \text{ etc.},$$

called *elements*, which are either constant, or of which the variations are known in advance.

x, y, z, etc., being functions of a, b, c, etc., we may express their relations to the latter in the form

$$x=f(a, \; b, \; c, \ldots \; t), \qquad \text{(27)}$$

with as many other equations as we have variables y, z, etc., to compute or observe. We then have problems of two classes:

I. From known or assumed values of the elements a, b, c, etc., to find the values of x, y, z, etc., at any time.

II. From a series of observed values of x, y, z, etc., to find the values of the elements.

If nearly correct values of the elements are known, we may compare the values of x, y, z, etc., computed from them with the observed values of those quantities. In investigating the relations in this way the elements are, in the language of mathematics, *independent variables*, while x, y, z, etc., are functions.

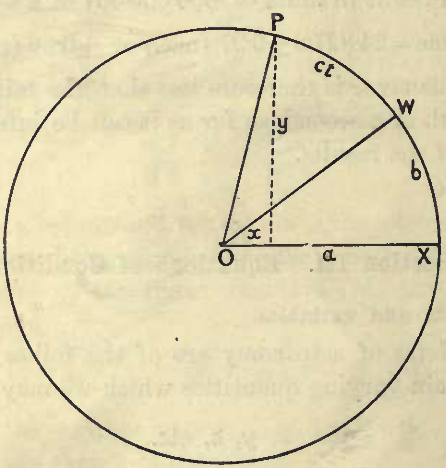


FIG. 3.

As an example, let us take the case of an object P, moving in a circle of radius a around a centre O with a uniform motion. If we put b for the angle XOW at a certain given moment from which we count the time, which moment we call the *epoch*, and c for the arc through which the object moves in unit of time, then the value of XOP at any time t after the epoch will be

$$b+ct,$$

and the rectangular coordinates of P will be

$$\left.\begin{array}{l} x=a\cos(b+ct) \\ y=a\sin(b+ct) \end{array}\right\} \quad \text{(28)}$$

If a, b, and c are given, we may compute x and y for as many epochs as we please by these equations.

Suppose now that we can observe or measure the coordinates x and y at certain moments t_1, t_2, etc., after the epoch. Then, if a, b, and c are known, we may, by substituting t_1, t_2, etc., for t in (28), compute x and y for the moments of observation. If the computed values agree with the observed values, well; if not, we have to investigate the cause of the discrepancy. This may be either errors in our measures of the coordinates, or errors in the values a, b, and c used in the computation. Possibly a third cause may have to be considered—error in the fundamental hypothesis of uniform circular motion of P; but we do not consider this at present.

Next take, as an extreme case, that in which the values of the elements a, b, and c are entirely unknown. Then we cannot compute (28) at all, for want of data. What we have to do is to reverse the process and determine a, b, and c from the observed values of x and y at the known times t_1, t_2, etc. If we call these observed values

$$x_1, \ y_1, \ x_2, \ y_2, \ \text{etc.},$$

we shall have to determine the values of a, b, and c from the system of equations

$$\left.\begin{array}{l} a\cos(b+ct_1)=x_1 \\ a\sin(b+ct_1)=y_1 \\ a\cos(b+ct_2)=x_2 \\ \dots\dots\dots\dots\dots\dots \end{array}\right\} \quad \dots\dots\dots\dots\dots\dots\dots\dots (29)$$

Here the second members of the equations are the observed values of x and y, while a, b, and c are the unknowns to be determined.

Equations of this kind are called *equations of condition*, because they express the conditions which the elements a, b, and c must satisfy in order that the results of computation with them may agree with observation.

Formally, the unknowns may be considered as determinable from a sufficient number of independent equations of the form (29). Usually such equations do not admit of solution except by tentative processes. But with three observed values of x and y at very different points on the circle we may derive approximate values of a, b, and c, which will form the basis for a further investigation.

35. Method of correcting provisional elements.

In most of the problems of astronomy, we do not regard the elements themselves as unknown quantities, but start with approximate values, supposed to be very near the truth, and take as unknowns the small corrections which we must add to these assumed or provisional values in order to get the true values. The corrections which these preliminary elements require are introduced by development in the following way:

Taking the general form (27), let

$$a_0,\ b_0,\ c_0, \ldots$$

be the provisional values of the elements and

$$\delta a,\ \delta b,\ \delta c,\ \ldots$$

the corrections which they require. Then the true but unknown values of the elements will be

$$\left.\begin{array}{l} a = a_0 + \delta a \\ b = b_0 + \delta b \\ c = c_0 + \delta c \\ \vdots \quad \vdots \quad \vdots \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

We substitute these values in (27) and develop by Taylor's theorem

$$\left.\begin{array}{l} x = f(a_0,\ b_0,\ c_0,\ t) \\ \quad + \dfrac{dx}{da_0}\delta a + \dfrac{dx}{db_0}\delta b + \dfrac{dx}{dc_0}\delta c + \ldots \\ \quad + \text{terms of the second and higher orders in } \delta a,\ \delta b, \text{ etc.} \end{array}\right\} (31)$$

From the nature of the case the provisional values are quite arbitrary, except that they should not deviate too widely from the truth. We are, therefore, free to choose their values so as to simplify the computation whenever this is practicable.

In practice we nearly always have to suppose the terms of the second and higher orders in (31) so small that they may be neglected. If such is not the case, it is commonly easier to repeat the computation with better values of the provisional elements than to consider the higher terms in question.

In the second member of (31) the first term is the value of x computed with the assumed values of the elements. Let us put

x comp.; the computed value.

x obs.; the observed value.

By taking this observed value as the first member of (31), dropping the third line of the equation and transposing, we have

$$\frac{dx}{da_0}\delta a+\frac{dx}{db_0}\delta b+\frac{dx}{dc_0}\delta c+\ldots=x\text{ obs.}-x\text{ comp.} \quad \text{.......(32)}$$

In this equation all the quantities are known numerically except δa, δb, and δc.

Example. The following coordinates of the satellite *Titania* of the planet Uranus, relative to the planet, are derived from observations by See at Washington in 1901:

	Time.	x.	y.	
(1)	May 13·5026	$-24''\cdot95$	$-22''\cdot05$	(33)
(2)	„ 15·5007	$+18\cdot61$	$-26\cdot85$	
(3)	„ 17·5008	$+29\cdot46$	$+15\cdot03$	
(4)	„ 22·5014	$-20\cdot04$	$-26\cdot67$	

Let us as a first hypothesis assume the motion in the apparent orbit to be circular and uniform. If we compute the polar coordinates, r (or a) and $\theta=b+ct$, from the above values of x and y for each of the four observations, by the usual formulae

$$r\cos\theta=x$$
$$r\sin\theta=y$$

we find the average value of r to be about $33''\cdot08$. Also by dividing the differences of the θ's by the elapsed intervals we find that the four values of θ may be closely represented by the hypothesis that

$$\left.\begin{array}{ll}\text{On May } 13\cdot5026, & \theta=221^\circ\ 28' \\ \text{Daily motion of } & \theta=c=\ \ 41\ \ \ 15\end{array}\right\} \quad \text{...........(34)}$$

We may take our initial epoch when we please; generally it is best to take it near the mean of all the times of observation, so that the sums of the positive and negative values of t shall nearly balance each other. For the first part of the computation, however, it will best serve our purpose to take a moment near the first observation, namely May 13·5, as the epoch. Our values of t_1, t_2 = etc. will then be found by subtracting this date from the others, and will be

$$t_1 = 0{\cdot}0026\,;\quad t_2 = 2{\cdot}0007\,;\quad t_3 = 4{\cdot}0008\,;\quad t_4 = 9{\cdot}0014.$$

From (34) we find

$$b_0 = 221^\circ\ 28' - t_1 c = 221^\circ\ 22'. \qquad (35)$$

We find the following values of r and θ from the measures of x and y:

	r.	$\theta = b + ct$.	Diff.
(1)	33″·30	221° 28′	83° 16′
(2)	32 ·66	304 44	82 18
(3)	33 ·07	27 2	206 3
(4)	33 ·36	233 5	

We know that, as a matter of fact, the apparent curve described by the satellite is slightly elliptical. But, for the purpose of illustration, we shall find how nearly the observations can be represented on the hypothesis of circular and uniform motion.

We therefore adopt these values of b_0 and c_0:

$$\left.\begin{aligned} b_0 &= 221^\circ\ 22' \\ c_0 &= \ \ 41\ \ 15 \end{aligned}\right\}, \qquad (36)$$

and we take

$$a_0 = 33''{\cdot}08.$$

We now have all the data for computing x and y from (28) or (29). The results, and the excess of each observed coordinate over that computed, are found to be as follows:

Dates.	$b+ct$.	x comp.	y comp.	Δx.	Δy.
(1)	221° 28′	−24″·79	−21·91	−0″·16	−0″·14
(2)	303 54	+18 ·45	−27·46	+0 ·16	+0 ·61
(3)	26 24	+29 ·63	+14·71	−0 ·17	+0 ·32
(4)	232 40	−20 ·06	−26·30	+0 ·02	−0 ·37

(37)

Here Δx and Δy are the excesses of the observed values of x and y given in (33) over the computed values.

Next we form the equations of condition for the corrections from (31). By differentiating (28), we have

$$\left.\begin{aligned} \frac{dx}{da} &= \cos(b+ct), & \frac{dy}{da} &= \sin(b+ct), \\ \frac{dx}{db} &= -a\sin(b+ct), & \frac{dy}{db} &= a\cos(b+ct), \\ \frac{dx}{dc} &= t\frac{dx}{db}, & \frac{dy}{dc} &= t\frac{dy}{db}. \end{aligned}\right\} \quad \text{.........(38)}$$

We now change our epoch at pleasure. In forming equations in which t enters, it is generally convenient to choose as the initial epoch a moment near the mean of all the times of observation. In the present case we shall have the simplest computation by taking the moment of the third observation as epoch. Then, dropping useless decimals, the values of t are -4, -2, 0, $+5$.

By using these four values of t in these equations and the values of a_0, b_0, c_0 in (36), we find four values of each coefficient, and eight equations of the form (32), four from x and four from y. These equations are

$$\left.\begin{array}{rrrrl} t=-4; & -0{\cdot}749\delta a & +21{\cdot}9\delta b & -\ 88\delta c & =-0''{\cdot}16 \\ -4; & -0{\cdot}662 & -24{\cdot}8 & +\ 99 & =-0\ {\cdot}14 \\ -2; & +0{\cdot}558 & +27{\cdot}5 & -\ 55 & =+0\ {\cdot}16 \\ -2; & -0{\cdot}830 & +18{\cdot}5 & -\ 37 & =+0\ {\cdot}61 \\ 0; & +0{\cdot}896 & -14{\cdot}7 & 0 & =-0\ {\cdot}17 \\ 0; & +0{\cdot}444 & +29{\cdot}6 & 0 & =+0\ {\cdot}32 \\ +5; & -0{\cdot}607 & +26{\cdot}3 & +132 & =+0\ {\cdot}02 \\ +5; & -0{\cdot}795 & -20{\cdot}1 & -100 & =-0\ {\cdot}37 \end{array}\right\} \quad \text{......(39)}$$

These eight equations have only three unknowns to be determined. We cannot satisfy them all with any values of the unknowns; but whatever values we adopt, there will be outstanding differences between the two members of the equations, which we should make as small as possible.

These differences are what we have in § 29 called *residuals*. They are functions of the unknown quantities, and we seek to determine the best values of the latter from the principle developed in § 27:

The best values of the unknown quantities which can be derived from a system of equations greater in number than the unknowns are those which make the sum of the squares of the residuals, multiplied by their respective weights, a minimum.

36. Conditional and normal equations.

We have to show the simple and elegant process by which values of the unknowns are found which reduce the function of the residuals above defined to a minimum. For this purpose let us consider the general case of a system of linear equations exceeding the unknown quantities in number. We consider the absolute terms or second members of the equations to be affected by a greater or less probable error, a judgment which we express by assigning to each such term a weight proportional to the inverse square of the probable error.

Let the conditional equations, with their weights, be

$$\left.\begin{array}{ll} a_1x+b_1y+c_1z+\ldots=n_1; & \text{weight}=w_1 \\ a_2x+b_2y+c_2z+\ldots=n_2; & \quad ,, \quad =w_2 \\ a_3x+b_3y+c_3z+\ldots=n_3; & \quad ,, \quad =w_3 \\ \vdots \quad \vdots \quad \vdots \quad\quad \vdots & \quad\quad\quad \vdots \end{array}\right\}, \quad \ldots\ldots\ldots(40)$$

of which the number is supposed to exceed that of the unknowns.

We also put $\pm r_1; \quad \pm r_2; \quad \pm r_3 \ldots \pm r_n$

for the residuals left when n_1, n_2, etc., are subtracted from the first members. Any one of the equations may then be written in the form

$$r_i=a_ix+b_iy+c_iz+\ldots-n_i. \quad \ldots\ldots\ldots\ldots\ldots\ldots(41)$$

This equation gives the r's as functions of the unknowns x, y, z, etc., and our problem is: What values of the unknowns will make the function

$$\Omega=w_1r_1^2+w_2r_2^2+\ldots+w_nr_n^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(42)$$

a minimum? The required conditions are that the derivatives of Ω as to x, y, and z, etc., shall vanish. We have

$$\frac{d\Omega}{dx}=\frac{d\Omega}{dr_1}\frac{dr_1}{dx}+\frac{d\Omega}{dr_2}\frac{dr_2}{dx}+\ldots=0, \quad \ldots\ldots\ldots\ldots\ldots(43)$$

with similar equations in y, z, etc. Also

$$\frac{d\Omega}{dr_i} = 2w_i r_i = 2w_i(a_i x + b_i y + \ldots - n_i),$$

and, from (41), $$\frac{dr_i}{dx} = a_i; \quad \frac{dr_i}{dy} = b_i, \text{ etc.}$$

Thus (43) becomes, when we divide by 2,

$$\begin{aligned} &w_1 a_1(a_1 x + b_1 y + c_1 z + \ldots) - w_1 a_1 n_1 \\ &+ w_2 a_2(a_2 x + b_2 y + c_2 z + \ldots) - w_2 a_2 n_2 \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0. \end{aligned}$$

In the same way, using y instead of x in (43),

$$\begin{aligned} &w_1 b_1(a_1 x + b_1 y + c_1 z + \ldots) - w_1 b_1 n_1 \\ &+ w_2 b_2(a_2 x + b_2 y + c_2 z + \ldots) - w_2 b_2 n_2 \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = 0. \end{aligned}$$

Continuing the process, we shall have a similar equation for each unknown quantity.

The equations may be expressed in a condensed form by putting

$$\left.\begin{aligned} [aa] &= w_1 a_1^2 + w_2 a_2^2 + \ldots + w_n a_n^2 \\ [ab] &= w_1 a_1 b_1 + w_2 a_2 b_2 + \ldots + w_n a_n b_n \\ [bb] &= w_1 b_1^2 + w_2 b_2^2 + \ldots + w_n b_n^2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [an] &= w_1 a_1 n_1 + w_2 a_2 n_2 + \ldots w_n a_n n_n \end{aligned}\right\} \ldots\ldots\ldots\ldots(44)$$

We shall thus have

$$\left.\begin{aligned} [aa]x + [ab]y + [ac]z + \ldots &= [an] \\ [ab]x + [bb]y + [bc]z + \ldots &= [bn] \\ [ac]x + [bc]y + [cc]z + \ldots &= [cn] \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad & \quad \vdots \end{aligned}\right\} \ldots\ldots\ldots\ldots(45)$$

These are called *normal equations.* The first, originally derived by differentiating Ω as to x, is called the *normal equation in x*, because it is the one which determines x, and so with the other unknown quantities.

The most convenient practical method of forming the normal equations is to write under each conditional equation, or rather under each set of its coefficients, the product of the coefficients into the weight of the equation.

Another method is to multiply all the terms in each equation by the square root of its weight, thus reducing all the absolute terms to weight 1.

In either case, instead of writing the unknown quantities after each coefficient, we write them once for all at the top of the column of coefficients, as shown in the scheme which follows. This scheme also shows the arrangement of the check against errors, which we apply by putting

$$s_i = a_i + b_i + \dots .$$

SCHEME OF CONDITIONAL EQUATIONS.

x	y	z		s	n	w
a_1	b_1	c_1	...	s_1	n_1	w_1
w_1a_1	w_1b_1	w_1c_1	...	w_1s_1	w_1n_1	
a_2	b_2	c_2	...	s_2	n_2	w_2
w_2a_2	w_2b_2	w_2c_2	...	w_2s_2	w_2n_2	
etc.	etc.	etc.	...	etc.	etc.	

(46)

We now take each a and multiply it into all the quantities in the line below it, writing the product in a horizontal line, thus:

$w_1a_1^2$	$w_1a_1b_1$	$w_1a_1c_1$	...	$w_1a_1s_1$	$w_1a_1n_1$
$w_2a_2^2$	$w_2a_2b_2$	$w_2a_2c_2$	...	$w_2a_2s_2$	$w_2a_2n_2$
⋮	⋮	⋮		⋮	⋮
$w_ma_m^2$	$w_ma_mb_m$	$w_ma_mc_m$	...	$w_ma_ms_m$	$w_ma_mn_m$
$[aa]$	$[ab]$	$[ac]$		$[as]$	$[an]$

The summation of the columns will then give the values of

$$[aa],\ [ab],\ [ac], \dots [an],$$

as also of

$$[as] = w_1a_1s_1 + w_2a_2s_2 + \dots .$$

We then proceed in the same way with the b's, multiplying each b_i; into the line of quantities

$$w_ib_i,\ w_ic_i, \dots w_is_i,\ w_in_i.$$

Adding the columns as before, we shall have the values of

$$[bb],\ [bc], \dots [bs],\ [bn].$$

It is not necessary to multiply the b's into the wa's, because the products have already been obtained by multiplying the a's into the wb's:

$$wa \times b = wb \times a$$

and

$$[ab] = [ba].$$

If the computation is correct, we should have

$$\left.\begin{array}{l} [as] = [aa] + [ab] + [ac] + \dots \\ [bs] = [ab] + [bb] + [bc] + \dots \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\ [sn] = [an] + [bn] + [cn] + \dots \end{array}\right\} \quad \dots\dots\dots\dots\dots (47)$$

In this way we find the coefficients of all the normal equations. Then, by solving the latter, we shall have those values of the unknown quantities which will make the sum of the squares of the errors into their weights, or the function Ω a minimum.

37. Solution of the normal equations.

In the usual computations of spherical astronomy, there are seldom more than three unknown quantities. A brief indication of the practical method of solution in this case will, therefore, suffice. We take the coefficients of the first normal equation:

$$[aa],\ [ab],\ [ac]\ \dots,$$

multiply them all by the successive quotients

$$\frac{[ab]}{[aa]},\ \frac{[ac]}{[aa]},\ \dots\ \frac{[as]}{[aa]},$$

and write the products under the coefficients of the other equations. The product $\frac{[ab]}{[aa]} \times [aa]$ will be $[ab]$ simply, and so need not be formed, unless as a test of the accuracy of the multiplier. We shall thus have pairs of equations, the first of each pair being the normal equation in one of the quantities y, z; the second the product of the equation in x by the appropriate factor, thus:

$$[ab],\ [bb],\ [bc],\ \dots\ [bs],\ [bn],$$

$$\frac{[ab][ab]}{[aa]},\ \frac{[ab][ac]}{[aa]},\ \dots\ \frac{[ab][as]}{[aa]},\ \frac{[ab][an]}{[aa]}.$$

Subtracting these from each other, we shall have an equation from which x is eliminated.

Then, applying the same process of eliminating x to the remaining normal equations, we shall have a set of equations between the unknowns y, z, etc.

Subjecting these equations to the same process, we shall reach a set of equations without x or y. Going on in the same way, we at length reach an equation with only one unknown quantity, say z of the form

$$Az = N,$$

which gives

$$z = \frac{N}{A}.$$

Then, by successive substitution in the equations previously formed, we obtain the values of the other unknown quantities.

Example. We may take as an example the equations (39), first subjecting them to a transformation. In the conditional equations it is always convenient to have the mean value of the coefficients of any one unknown not vastly different from those of the other unknowns. In (39) the coefficients of δc have a mean value about 100 times as large as those of δa and 30 times those of δb. We may avoid this inconvenience by using as unknown quantities

$$\left.\begin{aligned} x &= 0{\cdot}1\delta a, \ldots \delta a = 10x \\ y &= 3\delta b, \ \ldots \delta b = \tfrac{1}{3}y \\ z &= 10\delta c \ \ldots \delta c = 0{\cdot}1z \end{aligned}\right\} \qquad (48)$$

The substitution of these expressions will change the first equation into

$$7{\cdot}5x - 7{\cdot}3y + 8{\cdot}8z = 0''{\cdot}16.$$

Treating the other equations in the same way, and adding the three coefficients of each equation to form s the scheme is this:

No.	*a.*	*b.*	*c.*	*s.*	*n.*	*w.*
1	$-7{\cdot}5x$	$+7{\cdot}3y$	$-8{\cdot}8z$	$-9{\cdot}0$	$-0''{\cdot}16$	1
2	$-6{\cdot}6$	$-8{\cdot}3$	$+9{\cdot}9$	$-5{\cdot}0$	$-0\ {\cdot}14$	1
3	$+5{\cdot}6$	$+9{\cdot}2$	$-5{\cdot}5$	$+9{\cdot}3$	$+0\ {\cdot}16$	1·5
4	$-8{\cdot}3$	$+6{\cdot}2$	$-3{\cdot}7$	$-5{\cdot}8$	$+0\ {\cdot}61$	1·5
5	$+9{\cdot}0$	$-4{\cdot}9$	$0{\cdot}0$	$+4{\cdot}1$	$-0\ {\cdot}17$	1
6	$+4{\cdot}4$	$+9{\cdot}9$	$0{\cdot}0$	$+14{\cdot}3$	$+0\ {\cdot}32$	1
7	$-6{\cdot}1$	$+8{\cdot}8$	$+13{\cdot}2$	$+15{\cdot}9$	$+0\ {\cdot}02$	1
8	$-8{\cdot}0$	$-6{\cdot}7$	$-10{\cdot}0$	$-24{\cdot}7$	$-0\ {\cdot}37$	1

We have next to form the normal equations by (44). We multiply all the terms of the first equation by the first value of $wa = a$ (because $w=1$); then the terms of the second by the second value of wa, etc.

Dropping the last decimal figure of the product as unnecessary we thus find

aa.	*ab.*	*ac.*	*as.*	*an.*
56·2	−54·8	+66·0	+67·5	+1·20
43·6	+54·8	−65·3	+33·0	+0·92
47·1	+75·6	−46·2	+76·5	+1·35
103·4	−74·7	+46·1	+74·7	−7·59
81·0	−45·0	0·0	+36·0	−1·53
19·4	+44·0	0·0	+63·4	+1·41
37·2	−53·1	−80·5	−96·4	−0·12
62·4	+52·9	+79·0	+194·3	+2·92
450·3	−0·3	−0·9	+449·0	−1·44

The check against error is

$$[aa]+[ab]+[ac]=[as],$$

a condition which we find to be satisfied. Thus the first normal equation, or that in x, is

$$450x - 0{\cdot}3y - 0{\cdot}9z = -1''{\cdot}44,$$

the decimal being dropped from the coefficient of x because it is unnecessary.

We next multiply the coefficients by the respective values of wb, omitting the first, because we already have the products ab. We thus find

$$[bb]=543{\cdot}3;\quad [bc]=-72{\cdot}1;\quad [bs]=+471{\cdot}0;\quad [bn]=+14''{\cdot}34.$$

We apply the check

$$[ab]+[bb]+[bc]=[bs],$$

which comes out

$$470{\cdot}9=471{\cdot}0.$$

The error of 0·1 is less than the probable error from omitted decimals.

Multiplying by the coefficients wc, we find

$$[cc]=515{\cdot}4;\quad [cs]=+442{\cdot}6;\quad [cn]=-0''{\cdot}73.$$

The third check equation

$$[ac]+[bc]+[cc]=[cs]$$

comes out $442{\cdot}4=442{\cdot}6$,

which is as near as could be expected.

As a final check, we multiply each n by the corresponding value of ws, and add the eight products.

The result is $[sn]=+12''{\cdot}19$.

The check equation

$$[an]+[bn]+[cn]=[sn]$$

becomes $12''{\cdot}17=12''{\cdot}19$,

of which the error is as small as could be expected.

The normal equations to which we are thus led, omitting unnecessary decimals, are:

$$\left.\begin{array}{lllll} a. & b. & c. & s. & n. \\ 450x- & 0{\cdot}3y- & 0{\cdot}9z+ & 449= & -\ 1''{\cdot}44 \\ -0{\cdot}3+ & 543 & -\ 72{\cdot}1 & +471= & +14\ {\cdot}34 \\ -0{\cdot}9- & 72{\cdot}1 & +515 & +442= & -\ 0\ {\cdot}73 \end{array}\right\}\ \ldots\ldots\ldots(49)$$

It is unnecessary to write the coefficients to the left of the diagonal line $[aa]\ldots[ac]$; they are, however, given for completeness. The values of the sums $[as]\ldots[cs]$ are also written in, because they may be used as a check on the solution.

To proceed with the solution in the regular way, we should multiply the first equation by the factor $\frac{[ab]}{[aa]}$, and subtract the product from the second; then by the factor $\frac{[ac]}{[aa]}$, and subtract it from the third. Thus we eliminate x, and find the two equations

$$\left.\begin{array}{r} 543y-72{\cdot}1z=+14''{\cdot}34, \\ -72{\cdot}1y+515z=-\ 0\ {\cdot}73, \end{array}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots(50).$$

We next multiply the first of these equations by $-\frac{72{\cdot}1}{543{\cdot}4}$, giving

$$-72{\cdot}1y+10z=-1''{\cdot}90.$$

Subtracting this from the last equation, we eliminate y, and have

$$505z=1{\cdot}17.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(51)$$

Whence $z=+0{\cdot}00232.$ ....(52)

We now substitute this value of z in the first equation (50), and thus obtain y; then the values of y and z in the first equation (49) to obtain x. The results are

$$\left.\begin{array}{ll} x = -0''{\cdot}0032, & \delta a = -0''{\cdot}03, \\ y = +0\ {\cdot}0264, & \delta b = +0\ {\cdot}0088, \\ z = +0\ {\cdot}00232, & \delta c = +0\ {\cdot}000232. \end{array}\right\} \qquad (53)$$

From the way in which we have formed the differential coefficients

$$\frac{dx}{da}, \quad \frac{dx}{db}, \quad \frac{dx}{dc}, \ldots,$$

the value of δa comes out in seconds, and that of δb and δc in arc. We reduce the values of the latter to minutes by multiplying by 3438′, the minutes in the unit radius, and thus obtain

$$\delta b = +30'{\cdot}3,$$
$$\delta c = +\ 0{\cdot}8.$$

Applying these corrections to the adopted values of a, b, and c, we have, for their definitive values from all the observations,

$$a = 33''{\cdot}05,$$
$$b = 221^\circ\ 52'{\cdot}3,$$
$$c = \ 41^\circ\ 15'{\cdot}8.$$

The next step is to compute the values of x and y from these elements for the dates of the separate observations, being careful to use the precise values of t, rather than the approximate ones. Subtracting these x's and y's from the observed ones, we have the definitive residuals to be used in deriving the probable errors.

38. Weights of unknown quantities whose values are derived from equations of condition.

The theory of errors as developed in the preceding section applies only to the probable error of a directly observed quantity, and not to that of an element derived by the solution of equations of condition. The error of a result consequent upon an error of observation may be smaller or greater than the latter to any extent, the amount depending on the relation of the result to the observation. For example, if AB be a line, and C a distant object, whose distance from A or B is determined by measuring the angles at A and B, and then computing the sides AC and BC of the triangle; it is evident that, the farther away C is, the greater the effect of an error in the angles upon the computed distance. The effect of an error in B may be imagined by supposing that, while the line AB and the direction AC remain fixed, the side BC is allowed to turn on B as a pivot through an angle equal to the possible error. The corresponding motion of the point C will be the effect of this error on AC.

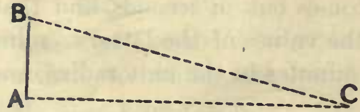


FIG. 4.

The great advantage of the method of least squares arises from the fact that it affords us a means of determining not only the values of the quantities sought, but the effect of the probable errors of observation upon those values. The theory upon which this subject rests is too extended for development in the present chapter; we must therefore confine ourselves to a statement of method. The two steps in the case are:

(1) The determination of the weight of the unknown quantity.

(2) The determination of the mean or probable error e_0 corresponding to weight unity.

The probable error of each quantity is then found by dividing e_0 by the square root of the weight.

There are different ways of finding the weights, of which the one most easily remembered is the following:

Having the normal equations in the form (45), we equate the absolute term of each equation to a literal quantity, say A, B, C, etc. The equations thus appear in the form

$$\left.\begin{array}{c}[aa]x+[ab]y+[ac]z+\ldots=[an]=A \\ [ab]x+[bb]y+[bc]z+\ldots=[bn]=B \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots\end{array}\right\} \quad \ldots\ldots\ldots(54)$$

We then carry through a simultaneous duplicate solution, one numerical, the other in terms of A, B, C, etc., as literal quantities. The final values of the unknown, will then be, not only their numerical values, but their expressions as linear functions of A, B, C, etc., in the form

$$x = \text{a number} = kA + lB + \dots$$
$$y = \quad \text{"} \quad = k'A + l'B + \dots,$$

where k, l, etc., will be numerical coefficients.

If the numerical work is correct, the values of x, y, etc., found by substituting the values (54) of A, B, C in these expressions, should be the same as those found by the numerical solution.

In these last expressions the diagonal coefficients k, l', etc., are the reciprocals of the weights of the corresponding quantities.

To find the probable errors from the discrepancies of the observations among themselves, we compute the residual r of each original equation of condition by substituting in it the values of the unknown quantities. We then form the sum

$$\Omega = w_1 r_1^2 + w_2 r_2^2 + w_3 r_3^2 +$$

and divide it by $n - m$, n being the number of equations and m that of the unknown quantities. The quotient is the square of the mean error for weight unity, which, being divided by the square root of the final weight of each quantity, gives its mean error.

39. Special case of a quantity varying uniformly with the time.

Let us apply the preceding results to the following case. We have a quantity x, of whose value we know or assume only that it varies uniformly with the time. We express this property by putting

t, the time, measured from an initial epoch;

z, the value of x at this epoch;

y, the increase of x in unit of time.

We shall then have, in general,

$$x = z + ty. \qquad (55)$$

When we know the values of z and y we can determine x at any time t by means of this equation.

Let it now be given that, at the several epochs

$$t_1,\ t_2,\ t_3, \dots t_n$$

we have observed the value of x to be

$$x_1,\ x_2,\ x_3, \dots x_n,$$

and that the problem is, from these observations, to derive values of z and y. We do this by equating the values of x in the form (55) to the observed values. For example, at the time t_1 we have for the value of x from (55)

$$x = z + t_1 y,$$

while the observed value is x_1. Equating these, we have

$$z + t_1 y = x_1,$$

where the second member is the observed x_1. Forming a similar equation from each of the other observations and adding the weights, we have

$$\left.\begin{array}{ll} z + t_1 y = x_1 & (\text{weight} = w_1) \\ z + t_2 y = x_2 & \quad ,, \quad = w_2 \\ \vdots \quad \vdots \quad \vdots & \quad ,, \quad \vdots \\ z + t_n y = x_n & \quad ,, \quad w_n \end{array}\right\} \dots\dots\dots\dots\dots\dots (56)$$

We thus have a system of equations of condition of which x and y are the unknown quantities to be determined.

If the observations were absolutely free from error, the values of z and y could be determined from any two of the equations. But, as all the observations are liable to error, let us put

$$r_1,\ r_2, \dots r_n$$

for the residual differences between the values of $x_1, x_2, \dots x_n$ as computed with any arbitrary values of z and y, and the observed values of $x_1, x_2, \dots x_n$. Then, instead of the equations (56) having the form as written, they will have the form

$$z + ty = x + r,$$

or transposing x to the first member, the system of equations will become

$$\left.\begin{array}{l} z + t_1 y - x_1 = r_1 \\ z + t_2 y - x_2 = r_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ z + t_n y - x_n = r_n \end{array}\right\} \dots\dots\dots\dots\dots\dots\dots\dots (57)$$

We now introduce the same requirement as in taking the mean, namely that the sum of the squares of the residuals multiplied by the weights, or the value of

$$\Omega = w_1 r_1^2 + w_2 r_2^2 + \ldots + w_n r_n^2, \qquad (58)$$

shall be the least possible. This requires that we shall have

$$w_1 r_1 dr_1 + w_2 r_2 dr_2 + \ldots + w_n r_n dr_n = 0.$$

We have, by differentiating (57),

$$\left.\begin{aligned} dr_1 &= dz + t_1 dy \\ dr_2 &= dz + t_2 dy \\ \vdots \quad & \quad \vdots \quad \vdots \\ dr_n &= dz + t_n dy \end{aligned}\right\} \qquad (59)$$

Multiplying these by the corresponding values of r in (57), the condition reduces to the form

$$A dz + B dy = 0, \qquad (60)$$

where

$$\begin{aligned} A &= w_1(z + t_1 y - x_1) \\ &\quad + w_2(z + t_2 y - x_2) \\ &\quad + \ldots\ldots\ldots, \\ B &= w_1 t_1(z + t_1 y - x_1) \\ &\quad w_2 t_2(z + t_2 y - x_2) \\ &\quad + \ldots\ldots\ldots \end{aligned}$$

In order that (60) may be satisfied for all values of dz and dy, we must have

$$A = 0, \quad B = 0.$$

These equations may be written in a condensed form by putting

$$W = w_1 + w_2 + \ldots + w_n,$$

the sum of all the weights:

$$\left.\begin{aligned} [t] &= w_1 t_1 \quad + w_2 t_2 \quad + \ldots + w_n t_n \\ [x] &= w_1 x_1 \quad + w_2 x_2 \quad + \ldots + w_n x_n \\ [tt] &= w_1 t_1^2 \quad + w_2 t_2^2 \quad + \ldots + w_n t_n^2 \\ [tx] &= w_1 t_1 x_1 + w_2 t_2 x_2 + \ldots + w_n t_n x_n \end{aligned}\right\} \qquad (61)$$

The equations $A = 0$, $B = 0$, then become

$$\left.\begin{aligned} Wz + [t]y &= [x] \\ [t]z + [tt]y &= [tx] \end{aligned}\right\} \qquad (62)$$

These are the *normal equations*. From them the values of z and y are derived:

$$\left.\begin{aligned} z&=\frac{[tt][x]-[t][tx]}{W[tt]-[t]^2} \\ y&=\frac{W[tx]-[t][x]}{W[tt]-[t]^2} \end{aligned}\right\} \quad \text{.....................(63)}$$

Having found the values of z and y, that of x may be found for any time t by the equation

$$x=z+ty. \quad \text{...............................(64)}$$

40. The mean epoch.

The epoch from which we count t is arbitrary. The computation is simplest when we take for this epoch the weighted mean of all the times of observation. These, counted from any arbitrary epoch, being as before, $t_1, t_2, t_3, \dots t_n$, the weighted mean of all the times will be

$$\left. t_m=\frac{w_1t_1+w_2t_2+\dots+w_nt_n}{W}\right\}. \quad \text{.................(65)}$$

Now, we can take t_m as the epoch quite as well as the original epoch. Putting $\tau_1, \tau_2, \dots \tau_n$ for the times counted from t_m, we have

$$\begin{aligned} \tau_1&=t_1-t_m, \\ \tau_2&=t_2-t_m, \\ &\vdots \end{aligned}$$

Then, putting z for the value of x at the mean epoch t_m, we have the equations of condition

$$\left.\begin{aligned} z+\tau_1y&=x_1 \\ z+\tau_2y&=x_2 \\ \vdots \quad \vdots \quad &\vdots \\ z+\tau_ny&=x_n \end{aligned}\right\} \quad \text{..........................(66)}$$

These equations are of the same form as (56), τ being put for t. Hence, by treating them as we did (56), we shall form normal equations like (62), except that τ takes the place of t. But, it is a fundamental property of the times τ counted from their mean epoch that their weighted mean must vanish. In fact

$$[\tau]=w_1\tau_1+w_2\tau_2+\dots+w_n\tau_n=0,$$

as the reader can readily show for himself by substituting the values of τ. Hence the equations (62) take the simple form

$$Wz=[x],$$
$$[\tau\tau]y=[\tau x],$$

and the solution is

$$\left.\begin{aligned} z&=\frac{[x]}{W} \\ y&=\frac{[\tau x]}{[\tau\tau]} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (67)$$

41. The probable errors of the unknown quantities may be determined in the following way. Expressing z and y as linear functions of $x_1, x_2, \ldots x_n$, § 24, Eq. 7 shows the mean error of z to be given by the equation

$$W^2\epsilon_z^2=w_1^2\epsilon_1^2+w_2^2\epsilon_2^2+\ldots+w_n^2\epsilon_n^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots (68)$$

$\epsilon_1, \epsilon_2, \ldots \epsilon_n$ being the respective mean errors of $x_1, x_2, \ldots x_n$. But, putting ϵ_0 for the probable error corresponding to weight 1, we have

$$w_i=\frac{\epsilon_0^2}{\epsilon_i^2},$$

whence $$w_i^2\epsilon_i^2=w_i\epsilon_0^2.$$

Thus, from (68), $$\epsilon_z^2=\frac{\epsilon_0^2}{W},$$

$$\epsilon_z=\pm\frac{\epsilon_0}{\sqrt{W}}.$$

This last equation expresses the conclusion: The probable error of the variable at the mean epoch is the same as if all the observations had been made at that epoch.

We have, in the same way, from (61) and (7),

$$\begin{aligned} \epsilon^2 \text{ of } [\tau x] &= w_1^2\tau_1^2\epsilon_1^2+w_2^2\tau_2^2\epsilon_2^2+\ldots w_n^2\tau_n^2\epsilon_n^2 \\ &=(w_1\tau_1^2+w_2\tau_2^2+\ldots+w_n\tau_n^2)\epsilon_0^2 \\ &=[\tau\tau]\epsilon_0^2. \end{aligned}$$

From (67) and (5), we see that the probable error of y is equal to that of $[\tau x]$ divided by $[\tau\tau]$. Hence

$$\epsilon_y=\pm\frac{\epsilon_0}{\sqrt{[\tau\tau]}}.$$

NOTES AND REFERENCES.

GAUSS's memoir, *Theoria Combinationis Observationum*, with its supplements, is a classic on this subject, dealing with it from the logical and mathematical point of view. A French translation by BERTRAND has been published, *Méthode des Moindres Carrés*, Paris, 1855.

The best practical method of arranging computations and deriving results is set forth in ENCKE's papers, *Ueber die Methode der Kleinsten Quadrate*, originally published as supplements to the *Berliner Jahrbuch* for 1834, 1835, 1836. These papers were subsequently reprinted in the collection, J. F. ENCKE's *Astronomische Abhandlungen*, three volumes, 8vo. Berlin, 1866.

WRIGHT, *Treatise on the Adjustment of Observations, with Applications to Geodetic and other Measures of Precision*, New York, 1884, is, as its title implies, prepared principally with a view to the problems of Geodesy; but the other applications are quite fully treated.

A similar treatise is MERRIMAN's *Text Book on the Method of Least Squares.*

HELMERT, *Die Ausgleichungsrechnung nach der Methode der Kleinsten Quadrate*, Leipzig, 1872, like Wright's work, has geodetic applications mainly in view.

KOLL, *Methode der Kleinsten Quadrate*, Berlin, 1893, is the most extended treatise with which the writer is acquainted. It deals almost entirely with practical applications, and the methods of forming, manipulating, and solving the equations.

CZUBER, *Theorie der Beobachtungsfehler*, Leipzig, 1891, presents a very elegant and attractive exposition of the entire subject from the mathematical standpoint.

Papers by J. W. L. GLAISHER in the *Monthly Notices R.A.S.*, vols. xl. and xli., deal with the forms of the determinants which implicitly enter into the solution, and are well worthy of study.

THIELE, T. N., *Theory of Observations*, London, Charles and Edwin Layton, 1903, is a very clear presentation of the subject of errors of observation.

Besides these, there are papers in the *Astronomical Journal* by JACOBI and others, and in the *Astronomische Nachrichten* by numerous writers, treating various aspects and phases of the subject.

The author of this Compendium hopes to publish a comprehensive work on the subject about the end of 1907.

PART II.

THE FUNDAMENTAL PRINCIPLES OF SPHERICAL ASTRONOMY.

CHAPTER IV.

SPHERICAL COORDINATES.

Section I. General Theory.

42. The positions of the heavenly bodies are defined by the values of coordinates by methods developed in analytic geometry of three dimensions. In astronomy the system of coordinates most used is a polar one, which, to distinguish it from that of polar coordinates in a plane, is commonly known as spherical. Rectangular coordinates are in frequent use in the computations of theoretical astronomy, but enter only incidentally into those of spherical astronomy. The fundamental elements of any system are:

1. An origin or point of reference. The points principally used in astronomy for this purpose are:

(*a*) A point of observation on the earth's surface at which an observer may be supposed located. Coordinates referred to this origin are called *apparent*.

(*b*) The centre of the earth. Coordinates referred to this origin are called *geocentric*.

(*c*) The centre of the sun. Coordinates referred to this origin are called *heliocentric*.

The position of a body is completely expressed when the direction and length of the line from the origin to the body are given. When the spherical system is used, the length of the line, called the *radius vector*, is one of the coordinates. The other two are commonly angles determining the direction of the radius vector.

The direction of the radius vector is expressed by its relation to a fundamental plane through the origin, which has, as its cognate or determining concept, a line through the origin perpendicular to it. It matters not whether the line determines the plane or the plane the line. In either case the line is taken for the axis of Z, and the plane is that of a system of coordinates X, Y.

In Fig. (5), let O be the origin.

P, the point whose position is to be defined.

OZ, the axis of Z.

OX, the adopted axis of X.

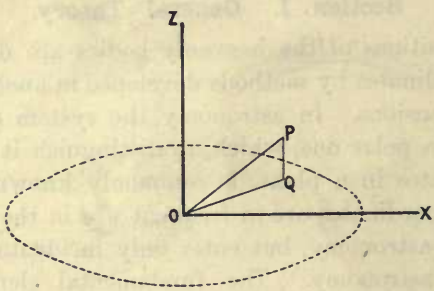


FIG. 5.

From P drop a perpendicular upon the fundamental plane, meeting the latter at the point Q. The angle XOQ, measured around O in a counter-clockwise direction, is one of those which determines the direction OP. It may be designated as the *longitude* of the point P, except when another designation is applied.

The other angle is either QOP, that which the radius vector makes with the fundamental plane, or ZOP, the angle which it makes with the fundamental axis. These angles are the complements of each other.

The angle QOP is, in the absence of a special designation, called the *latitude* of P. It is positive or negative according to whether P is situated on the side of the fundamental plane toward the positive direction of the axis of Z, or on the opposite side. The complementary angle ZOP is called the *polar distance* of P.

We readily see that the latitude is contained between the limits $+90°$ and $-90°$. The polar distance, connected with the latitude by the relation

$$\text{Polar distance} + \text{latitude} = 90°$$

is always positive, and varies between the limits 0° and 180°.

43. The radius vector, longitude, and latitude of a heavenly body being given, its position is uniquely determined by the following geometric construction. Pass a sphere round the origin as centre with a radius equal to the given radius vector. Pass through the axis of Z a plane making with the plane XOZ a dihedral angle equal to the given longitude. In this plane measure an angle ZOP equal to the given polar distance. Then the intersection of the line OP with the sphere will be the position of P, which will thus be completely and uniquely determined.

From the preceding definitions it will be seen that the longitude ranges between 0° and 360°. We may, if we choose, use negative longitudes, implying the measurement from OX in the negative or clockwise direction, This is frequently convenient when the longitude exceeds 270°.

44. The celestial sphere.

It is shown in spherical trigonometry that we may assist our conceptions of the lines, planes, and angles which enter into a trihedral angle by imagining a sphere having its centre at the vertex of the angle, and marking upon its surface the points and circular arcs in which the edges and planes of the trihedral angle intersect it. The parts of the trihedral angle are thus represented by the six parts of a spherical triangle.

A similar help is invoked in astronomy by introducing the conception of the celestial sphere, upon which we may conceive points to be marked and circles to be drawn. This sphere has its visible representation in the sky, and we may conceive points and circles upon it as marked or drawn on the sky.

It is common to consider the celestial sphere as of infinite radius. Then every point with which we are concerned may be regarded as situated at its centre. All lines parallel to each other intersect it in coincident points, and all planes parallel to each other in coincident great circles.

We may also conceive a finite sphere of any size to be drawn around the point of reference or origin of coordinates. There will then be a separate sphere for each separate origin. So far as results are concerned, it is indifferent which system is adopted in thought; but the conception of an infinite sphere is the simpler.

In this case, the direction of every line R in space is represented by the point Pr at which it intersects the sphere, and the direction of every plane L by the great circle Lc in which it cuts the sphere.

If P is perpendicular to L, Pr is the pole of Lc.

The angle between any two lines R and R' is measured by the circular arc between the points Pr and $P'r$, where they intersect the sphere.

The dihedral angle between two planes is equal to that at which the corresponding great circles intersect.

Correlative with the conception of a system of planes containing a line R is that of a system of great circles passing through the point Rc of the sphere. The great circle Lc having Pr as its pole then intersects the system at right angles. The circles which form the system are then called *secondaries to Lc.*

Small circles parallel to a great one are called *parallels*, and may be designated either as so related to the great circle, or by any point through which they pass.

It is shown in Fig. 6 how the coordinates which determine P as seen in Fig. 5 are represented on the sphere. X, Y, and Z, the axis of Y being added to the system of Fig. 5, are the points in which the rectangular axes intersect the sphere. They form the vertices of a triangular rectangular spherical triangle. The great circle AYX is that in which the fundamental plane of XY intersects the sphere. P is the point of intersection of the radius vector of a heavenly body with the sphere. ZPR is a quadrant from Z through P to R. The angle POQ, which we have called the latitude of the body, is now represented by the arc PR. The complementary arc ZP represents the polar distance. The arc XR represents the angle XOR, which we have called the longitude of the body. It may be equally represented

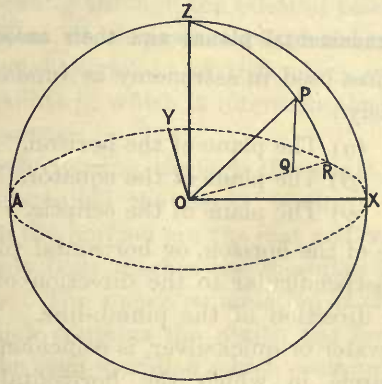


FIG. 6.

by the angle XZR at the vertex Z or by the dihedral angle between the planes XOZ and ROZ.

The point Z in which the fundamental axis intersects the sphere is called the pole.

The polar distance is, in the abstract, a more convenient coordinate than the latitude, because it is always positive. Its use thus avoids the danger of a mistake from assigning a wrong algebraic sign, which is incident to the use of latitudes. Polar distances have been used at the Greenwich Observatory since 1835, but in astronomical literature the latitudinal form is generally used.

We shall frequently use the notation

λ, the longitude,

β, the latitude.

A feature in which the longitudinal coordinate λ differs from the latitudinal one, β, is in the effect of small changes in its value upon the position of the point indicated. A change $\Delta\beta$ in β

always produces an equal change in the apparent position of the body, as seen from the origin. But, owing to the measures of λ being made around the pole as an axis, the apparent displacement due to a given $\Delta\lambda$ is less, the nearer the direction of the point P is to that of the pole, the general law being

$$\text{Displacement} = \cos\beta\Delta\lambda.$$

45. Special fundamental planes and their associated concepts.

The planes most used in astronomy as fundamental are three in number, namely:

(α) The plane of the horizon.
(β) The plane of the equator.
(γ) The plane of the ecliptic.

(α) The plane of the horizon, or horizontal plane, is defined as that which is perpendicular to the direction of gravity at any place, or to the direction of the plumb-line. The surface of a still liquid, say water or quicksilver, is coincident with this plane.

The great circle in which the horizontal plane cuts the celestial sphere is called the *celestial horizon*. The poles of the horizon are the points at which the vertical line intersects the sphere. That in the upward direction is called the *zenith*, that in the lower direction the *nadir*.

A distinction is sometimes made between the horizontal plane passing through the position of the observer, and the parallel plane passing through the centre of the earth. The first is called the *apparent*; the second, the *rational* or *geocentric* horizon. The distinction is unnecessary in the horizon on the infinite sphere, because the two planes cut the sphere in coincident circles.

Planes containing a vertical line are called *vertical planes*, and the corresponding circles of the sphere *vertical circles*. All vertical circles pass through the zenith and nadir, and are secondary to the celestial horizon.

A parallel to the horizon is called an *almucantur*.

(β) The plane of the equator is that which passes through the earth's centre at right angles to its axis of rotation. The circle in which it intersects the earth's surface is the *terrestrial* *equator.* The great circle in which it intersects the celestial sphere is called the *celestial equator.*

The poles of the celestial equator are the points in which the axis of rotation intersects the sphere. They are called the *celestial poles,* and are distinguished as north and south.

Great circles passing through the celestial poles are secondary to the equator. That which passes through the zenith of a place is called the *celestial meridian* of that place, or the *meridian* simply, and the points in which it intersects the horizon are the north and south points.

The vertical circle passing through the zenith at right angles to the meridian is called the *prime vertical.* The points in which it intersects the horizon are the east and west points.

When a heavenly body reaches the meridian of a place, it is said to *culminate.* The upper culmination is that through the semi-meridian which contains the zenith, the lower culmination is that through the semi-meridian which contains the nadir. In astronomical nomenclature it is common to indicate a lower culmination by the letters S.P., an abbreviation of *sub polo*; the upper one by U.C.

(γ) The plane of the ecliptic is that in which the earth moves around the sun, allowance being made for slight motions of the earth's centre perpendicular to it, and caused by the action of the moon and planets.

The great circle in which the plane of the ecliptic intersects the celestial sphere is called the *ecliptic.* The apparent course which the sun, to the observer on the earth, appears to describe around the celestial sphere in the course of a year, is practically coincident with the ecliptic, and is commonly used to define it.

Being great circles, the ecliptic and celestial equator intersect at two opposite points. These points are called the *equinoxes.* That point at which the sun apparently crosses the celestial equator, moving toward the north, is called the *vernal equinox*; the opposite point at which it crosses toward the south is called the *autumnal equinox.*

The angle at which the equator and ecliptic intersect is called the *obliquity of the ecliptic.* It is equal to the dihedral angle between the planes of the ecliptic and equator, and to the arc of the sphere between the poles of the ecliptic and of the equator.

The two opposite points on the ecliptic 90° from either equinox are termed the *solstices*, and are the points where the sun reaches its greatest angular distance from the equator, north or south.

Two great circles, secondaries to the equator, and at right angles to each other, pass—the one through the celestial poles and equinox, the other through the poles and the solstices. These are called *colures*. That which contains the equinoxes is called the *equinoctial colure*; and that which contains the solstices the *solstitial colure*.

The poles of the ecliptic lie on the solstitial colure, and it may be useful to remember that the north pole of the ecliptic is in 270°, or 18 hours of R.A. In middle northern latitudes it is at a greater or less distance north of the zenith in the early evenings of autumn, and is situated in the constellation Draco. The nearest conspicuous star is ω Draconis of the 4th or 5th magnitude, about 3° distant from it.

Conversely, the north celestial pole is in 90° of longitude, and is near the star α Ursae Minoris, hence called *Stella Polaris* commonly abbreviated to *Polaris*, or the *Pole star*. At present the distance is about 1° 12′. The distance is diminishing through precession, in consequence of which the pole will continually approach the star during the next two centuries, passing it in the year 2102 at a distance of about half a degree.

46. Special systems of coordinates.

Four systems of spherical coordinates are used in astronomy, each having one of the three fundamental planes just described as its plane of reference.

First system: altitude and azimuth. This has the horizon as its fundamental plane. The spherical coordinates which determine the direction of a heavenly body referred to this system are altitude and azimuth. The *altitude* is the vertical coordinate, and is the angle which the line drawn to the body makes with the plane of the horizon, or, on the sphere, it is the arc of the vertical circle through the body P, contained between

P and the horizon. The *zenith distance* is the complementary distance from the zenith to P.

The *azimuth* of P is the longitudinal coordinate, and is the arc of the horizon intercepted between the vertical circle through it and the north or south point.

In accordance with the general system, the positive direction in which azimuth is measured should be from the north point of the horizon through west, south, and east. But, in practice it is measured either from the north or the south point, and in either direction, east or west.

Second system: right ascension and declination. Here the axis of z is the rotation axis of the earth, and the fundamental plane is that of the equator.

The latitudinal coordinate is the angle which the radius vector of a heavenly body makes with the plane of the equator, and is called the *declination* of the body. The complementary angle which it makes with the axis of the earth is called the *north polar distance* of the body. When the centre of the earth is taken as the origin, the adjective *geocentric* is applied to the declination; when a point on the surface, the declination and polar distance are called *apparent*.

Through the position of a body and the two poles pass a semicircle. The angle which this semicircle makes with the equinoctial colure, measured from west toward east, is the longitudinal coordinate, and is called the *right ascension* of the body. We use the abbreviations:

R.A. = Right Ascension.

Dec. = Declination.

N.P.D. = North Polar Distance.

Positions on the surface of the earth are referred to the equatorial system. The astronomical *latitude* of a place is defined as the angle which the plumb-line at that place makes with the plane of the equator. Since this line, in the upward direction, marks the zenith, it follows that the declination of the zenith is equal to the latitude of the place. A corollary readily seen is that the altitude of the pole, and the zenith distance of the point at which the celestial equator intersects the meridian, are both equal to the latitude of the place. The complement of the latitude is for brevity called the *colatitude.* This is equal to the zenith distance of the pole and to the altitude of the point of intersection of the equator and meridian.

In this common system terrestrial longitude corresponds to Right Ascension, since both are measured in the plane of the equator or around the pole.

Declination being measured north from the celestial equator, and the meridian zenith distance of the equator being equal to the latitude, it follows that the zenith distance of any object on the meridian is equal to the latitude of the place minus the declination.

Putting, as usual in astronomy,

ϕ = Astronomical latitude,
z = Zenith distance south,
δ = Declination,

we have the relation $$z=\phi-\delta.$$

In astronomical practice is used:

α, the Right Ascension;
δ, the Declination.

Third system: declination and hour angle. Here the axis of Z is still that of rotation of the earth. But the semicircles from pole to pole are conceived to rotate with the earth and, therefore, to be apparently fixed. They are called *hour circles.* Evidently the celestial meridian, as already defined, is the hour circle through the zenith.

The angle which the hour circle through a body makes with the meridian is called the *hour angle* of the body. It is continually increasing toward the west owing to the rotation of the earth. Hence, for convenience, it is taken positively toward the west, though this is contrary to the mathematical convention we have described.

The *parallactic angle* is the angle between the hour circle and the vertical circle through a body.

Fourth system: longitude and latitude. The fourth system of axes has the ecliptic as its fundamental plane.

The *latitude* of a heavenly body is the angle which the line drawn to it from the origin makes with the ecliptic. The complementary angle which this line makes with that to the north pole of the ecliptic is called the *ecliptic polar distance* of the body.

Of the secondaries through the poles of the ecliptic one passes through the vernal equinox; this is taken as the initial circle ZX, Fig. 6. Another secondary, at right angles to this, is the solstitial colure.

The *longitude* of a heavenly body is the angle which the semicircle through it and the pole of the ecliptic makes with the initial circle through the vernal equinox.

47. Relations of spherical and rectangular coordinates.

To every system of spherical coordinates corresponds a rectangular system having the same origin and the same fundamental plane. The relation may be seen in § 43, Fig. 6. The axis of X is that passing in the fundamental plane from the origin to the initial point from which the longitudinal coordinate of the spherical system is measured. The axis of Y is in the same plane, perpendicular to X, its positive direction being toward the point of which the longitudinal coordinate is 90°. The axis of Z is that perpendicular to the fundamental plane, its positive direction being on the positive side of the plane.

The rectangular systems most in use correspond to the second and fourth of the spherical systems just described. That having the equator as its fundamental plane is called the *equatorial system*; that having the ecliptic the *ecliptical system.* These two systems have a common X-axis, directed toward the vernal equinox. The Z-axis of the equatorial system intersects the sphere at the celestial pole; that of the ecliptical system at the pole of the ecliptic. The Y-axis of the equatorial system intersects the celestial equator in 90° of R.A.; that of the ecliptical system intersects the ecliptic in 90° of longitude. Both of these Y-points are on the solstitial colure Zy, Fig. 7.

The natural origin of the equatorial system is the centre of the earth, and that of the ecliptical system the centre of the sun. But we may transfer either origin to any point in space—the centre of a planet or of a star for example, the direction of the axes remaining unchanged. The points in which the axes intersect the infinite celestial sphere will then remain unchanged also.

If we put, for the spherical coordinates and radius vector

$\lambda =$ the longitudinal coordinate,
$\beta =$ the latitudinal ,,
$r =$ the radius vector,

we see from the construction of Fig. 6 that the corresponding rectangular coordinates in any system will be given by the equations

$$\left.\begin{aligned} x &= r\cos\beta\cos\lambda \\ y &= r\cos\beta\sin\lambda \\ z &= r\sin\beta \end{aligned}\right\} \qquad (1)$$

In the equatorial system, which is the usual one in astronomy, the expressions are

$$\left.\begin{aligned} x &= r\cos\delta\cos\alpha \\ y &= r\cos\delta\sin\alpha \\ z &= r\sin\delta \end{aligned}\right\} \qquad (2)$$

48. Differentials of the rectangular and spherical coordinates.

The differentials of x, y, z, in the general system, are

$$dx = \cos\beta\cos\lambda\, dr - r\sin\beta\cos\lambda\, d\beta - r\cos\beta\sin\lambda\, d\lambda,$$
$$dy = \cos\beta\sin\lambda\, dr - r\sin\beta\sin\lambda\, d\beta + r\cos\beta\cos\lambda\, d\lambda,$$
$$dz = \sin\beta\, dr + r\cos\beta\, d\beta,$$

or

$$\left.\begin{aligned} dx &= \frac{x}{r}dr - z\cos\lambda\, d\beta - y\, d\lambda \\ dy &= \frac{y}{r}dr - z\sin\lambda\, d\beta + x\, d\lambda \\ dz &= \frac{z}{r}dr + r\cos\beta\, d\beta \end{aligned}\right\} \qquad (3)$$

The inverse expressions for the differentials of the polar in terms of the rectangular coordinates are found by multiplying these three equations in order by the coefficients in order of any one of the three differentials, and adding. Thus to express $d\rho$ we multiply by $\frac{x}{r}$, $\frac{y}{r}$, and $\frac{z}{r}$. To express $d\beta$ we multiply by $-z\cos\lambda$, $-z\sin\lambda$, and $r\cos\beta$. For dx we have only to multiply by $-y$ and x, noting that

$$x\sin\lambda - y\cos\lambda = 0.$$

We thus have

$$\left.\begin{aligned} dr &= \frac{x}{\rho}dx + \frac{y}{\rho}dy + \frac{z}{\rho}dz \\ &= \cos\beta\cos\lambda\, dx + \cos\beta\sin\lambda\, dy + \sin\beta\, dz \\ r^2 d\beta &= -z\cos\lambda\, dx - z\sin\lambda\, dy + r\cos\beta\, dz \end{aligned}\right\}, \quad \ldots\ldots\ldots(4)$$

or $$r\,d\beta = -\sin\beta\cos\lambda\, dx - \sin\beta\sin\lambda\, dy + \cos\beta\, dz,$$

$$r^2\cos^2\beta\, d\lambda = -y\,dx + x\,dy,$$

or $$r\cos\beta\, d\lambda = -\sin\lambda\, dx + \cos\lambda\, dy.$$

To form the expressions for the special case of the equatorial system we replace λ and β by α and δ, thus obtaining

$$\left.\begin{aligned} \cos\delta\, d\alpha &= \cos\alpha\frac{dy}{r} - \sin\alpha\frac{dx}{r} \\ d\delta &= -\sin\delta\cos\alpha\frac{dx}{r} - \sin\delta\sin\alpha\frac{dy}{r} + \cos\delta\frac{dz}{r} \end{aligned}\right\}. \quad \ldots.(4a)$$

We retain the cosines of β and δ as factors of $d\lambda$ and $d\alpha$ in order that the displacements represented by the products may represent arcs of a great circle on the sphere. As already remarked, the amount of displacement represented by a given value of $d\lambda$ and $d\alpha$ increases indefinitely as the pole is approached.

49. Relations of the equatorial and ecliptic coordinates.

The relations of these two systems may be seen in Fig. 7. Here Xy is the equator, XY the ecliptic intersecting it at X, the vernal equinox, which represents the common X-axis of the two systems. X is also the pole of the solstitial colure which is

$ZzYy$. The obliquity is also represented by either of the arcs yY, zZ, or by the angle zXZ.

Let us now put

x, y, z, the coordinates of a body referred to the equatorial system;

X, Y, Z, the coordinates of the same body referred to the ecliptical system;

ϵ, the obliquity of the ecliptic;

(x, X), (y, Y), (z, Z), etc., the angles between the several axes.

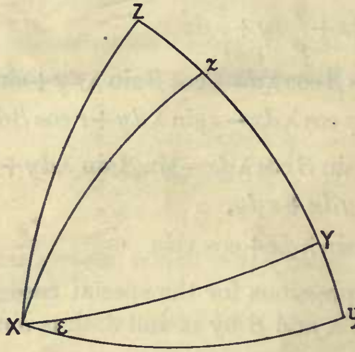


FIG. 7.

From the figure we readily see that

$$(x, X)=0,$$
$$(x, Y)=(x, z)=90^\circ,$$
$$(y, x)=(z, X)=90^\circ,$$
$$(y, Y)=(z, Z)=\epsilon,$$
$$(y, Z)=90^\circ+\epsilon,$$
$$(z, Y)=90^\circ-\epsilon.$$

The well known formulae of transformation are:

$$\left.\begin{aligned} X&=x\cos(x, X)+y\cos(y, X)+z\cos(z, X) \\ Y&=x\cos(x, Y)+y\cos(y, Y)+z\cos(z, Y) \\ Z&=x\cos(x, Z)+y\cos(y, Z)+z\cos(z, Z) \end{aligned}\right\}, \quad \text{........(5)}$$

and, conversely,

$$\left.\begin{aligned} x &= X\cos(x, X) + Y\cos(x, Y) + Z\cos(x, Z) \\ y &= X\cos(y, X) + Y\cos(y, Y) + Z\cos(y, Z) \\ z &= X\cos(z, X) + Y\cos(z, Y) + Z\cos(z, Z) \end{aligned}\right\} \quad \ldots\ldots\ldots(6)$$

We have thus the formulae of transformation

$$\left.\begin{aligned} X &= x \\ Y &= \quad y\cos\epsilon + z\sin\epsilon \\ Z &= -y\sin\epsilon + z\cos\epsilon \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

and, conversely,

$$\left.\begin{aligned} x &= X \\ y &= Y\cos\epsilon - Z\sin\epsilon \\ z &= Y\sin\epsilon + Z\cos\epsilon \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Section II. Problems and Applications of the Theory of Spherical Coordinates.

50. Right Ascension is almost universally expressed in time—hours, minutes, and seconds—instead of degrees, etc. The reason of this practice is that R.A. is determined by means of the sidereal time, on a system set forth in the next chapter.

Time and arc are mutually converted by multiplying or dividing by 15. A table for readily effecting this multiplication or division is found in Appendix II.

Tables of logarithms of the trigonometric functions with the argument in time have been published, but are not in general use. When not at hand, it is always easy to make the required conversion of the R.A. into arc. The principal applications of spherical astronomy into which time does not enter may be stated in the form of the solution of problems.

51. Problem I. *To convert longitude and latitude into right ascension and declination and vice versa.*

The formulae of conversion are readily derived from those for the transformation of rectangular coordinates. If in (8) we substitute for X, Y, and Z the expressions of x, y, and z (1), and for x, y, and z the corresponding values (2) in α and δ, r divides out, and we have

$$\left.\begin{aligned} \cos\delta\cos\alpha &= \cos\beta\cos\lambda \\ \cos\delta\sin\alpha &= \cos\epsilon\cos\beta\sin\lambda - \sin\epsilon\sin\beta \\ \sin\delta &= \sin\epsilon\cos\beta\sin\lambda + \cos\epsilon\sin\beta \end{aligned}\right\} \quad \ldots\ldots\ldots(9)$$

In the same way, from (7),

$$\left.\begin{aligned} \cos\beta\cos\lambda &= \cos\alpha\cos\delta \\ \cos\beta\sin\lambda &= \cos\epsilon\cos\delta\sin\alpha + \sin\epsilon\sin\delta \\ \sin\beta &= -\sin\epsilon\cos\delta\sin\alpha + \cos\epsilon\sin\delta \end{aligned}\right\} \quad \ldots\ldots\ldots(10)$$

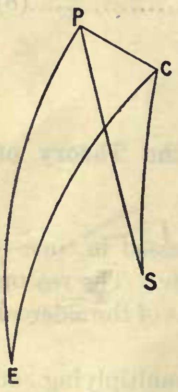


FIG. 8.

These equations are those among the parts of a spherical triangle. This triangle is that whose vertices are the two poles and the body. The geometric relations involved in the problem will be better seen by deriving them from this triangle.

Let P and C* be the respective poles of the equator and ecliptic, corresponding to z and Z in Fig. 7, and S the direction of the star. Let E be the vernal equinox which, being on each of the fundamental great circles, is 90° from either pole.

PCE is then a birectangular spherical triangle, in which CP is the obliquity, ϵ.

We also have

Angle $ECS = \lambda$, the longitude of S taken negatively.

Angle $EPS = \alpha$, the R.A. of S „ „

Side $CS = 90° - \beta$, the ecliptic N.P.D. of S.

Side $PS = 90° - \delta$, the N.P.D. of S.

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*The relative situation of the two poles and equinox in this figure is the obverse of that in Figure 7, so as to show it as we actually see it in looking up at the sky. In the preceding figures the celestial sphere has been represented as if seen from the outside, in order to show more clearly the geometric relations involved.

Hence, in the triangle PCS,

$$\left.\begin{array}{ll}\text{angle } P = 90° + \alpha & \equiv B \\ \text{angle } C = 90° - \lambda & \equiv A \\ \text{angle } S = & \equiv C \\ \text{side } PC = \epsilon & \equiv c \\ \text{side } PS = 90° - \delta & \equiv a \\ \text{side } CS = 90° - \beta & \equiv b\end{array}\right\} \qquad (11)$$

We add the usual symbols for the sides and opposite angles in order to facilitate writing the fundamental relations between the parts, which give the equations (9) and (10).

It is useful to note that one set of relations may be derived from the other by interchanging λ with α and β with δ, and changing ϵ into $-\epsilon$.

The numerical solution of the equations (9) will give $\sin\delta$ and $\cos\delta$ separately, the agreement of which will serve as a partial check on the accuracy of the computation. To adapt the formulae to logarithmic computation, we compute the auxiliaries m and M thus:

$$\left.\begin{array}{l} m \sin M = \sin\beta \\ m \cos M = \cos\beta \sin\lambda \end{array}\right\} \qquad (12)$$

Then

$$\left.\begin{array}{r} \sin\delta = m\sin(M+\epsilon) \\ \cos\delta\sin\alpha = m\cos(M+\epsilon) \\ \cos\delta\cos\alpha = \cos\beta\cos\lambda \end{array}\right\} \qquad (13)$$

Note that in these equations

$m = ES$ (distance of S from Equinox),

M = angle which ES makes with the Ecliptic,

$M+\epsilon$ = angle which ES makes with the Equator.

In the inverse solution we may compute

$$\left.\begin{array}{l} m \sin N = \sin\delta \\ m \cos N = \cos\delta \sin\alpha \end{array}\right\} \qquad (14)$$

Then

$$\left.\begin{array}{r} \sin\beta = m\sin(N-\epsilon) \\ \cos\beta\sin\lambda = m\cos(N-\epsilon) \\ \cos\beta\cos\lambda = \cos\delta\cos\alpha \end{array}\right\}, \qquad (15)$$

which we may use to compute β and λ.

The computation may be made yet shorter, thus: from the equations (12) we have

$$\tan M = \frac{\tan\beta}{\sin\lambda}, \quad\text{.........................}(16)$$

by which we compute M, m being omitted. To find α we take the quotient of the last two equations (13), substituting for m its value $\cos\beta\sin\lambda \div \cos M$ from the second of (12). Thus we have

$$\tan\alpha = \frac{\cos(M+\epsilon)\tan\lambda}{\cos M}. \quad\text{.....................}(17)$$

The quotient of the first two of (13) then gives

$$\tan\delta = \sin\alpha\tan(M+\epsilon). \quad\text{.....................}(18)$$

Also corresponding to (16)–(18) we have the equations

$$\left.\begin{aligned} \tan N &= \frac{\tan\delta}{\sin\alpha} \\ \tan\lambda &= \frac{\cos(N-\epsilon)\tan\alpha}{\cos N} \\ \tan\beta &= \sin\lambda\tan(N-\epsilon) \end{aligned}\right\}. \quad\text{...................}(19)$$

But this abbreviated method may fail to give an accurate result if α or λ is very near 0° or 180°, as the result may then come out as the quotient of two small quantities.

52. Use of the Gaussian equations.

The Gaussian equations for the spherical triangle may also be used with advantage in cases where the angle S of the triangle CPS is required, and, in any case, are rendered attractive in use by their elegance in form. Instead of S, its complement E is used: $E = 90° - S$. They are as follows:

ECLIPTICAL TO EQUATORIAL COORDINATES.

$$\left.\begin{aligned} \sin(45° - \tfrac{1}{2}\delta)\sin\tfrac{1}{2}(E+\alpha) &= \sin(45° + \tfrac{1}{2}\lambda)\sin(45° - \tfrac{1}{2}(\epsilon+\beta)) \\ \sin(45° - \tfrac{1}{2}\delta)\cos\tfrac{1}{2}(E+\alpha) &= \cos(45° + \tfrac{1}{2}\lambda)\cos(45° - \tfrac{1}{2}(\epsilon-\beta)) \\ \cos(45° - \tfrac{1}{2}\delta)\sin\tfrac{1}{2}(E-\alpha) &= \cos(45° + \tfrac{1}{2}\lambda)\sin(45° - \tfrac{1}{2}(\epsilon-\beta)) \\ \cos(45° - \tfrac{1}{2}\delta)\cos\tfrac{1}{2}(E-\alpha) &= \sin(45° + \tfrac{1}{2}\lambda)\cos(45° - \tfrac{1}{2}(\epsilon+\beta)) \end{aligned}\right\}. \quad (20)$$

EQUATORIAL TO ECLIPTICAL COORDINATES.

$$\left.\begin{aligned}\sin(45^\circ-\tfrac{1}{2}\beta)\sin\tfrac{1}{2}(E-\lambda)&=\cos(45^\circ+\tfrac{1}{2}\alpha)\sin(45^\circ-\tfrac{1}{2}(\epsilon+\delta))\\ \sin(45^\circ-\tfrac{1}{2}\beta)\cos\tfrac{1}{2}(E-\lambda)&=\sin(45^\circ+\tfrac{1}{2}\alpha)\cos(45^\circ-\tfrac{1}{2}(\epsilon-\delta))\\ \cos(45^\circ-\tfrac{1}{2}\beta)\sin\tfrac{1}{2}(E+\lambda)&=\sin(45^\circ+\tfrac{1}{2}\alpha)\sin(45^\circ-\tfrac{1}{2}(\epsilon-\delta))\\ \cos(45^\circ-\tfrac{1}{2}\beta)\cos\tfrac{1}{2}(E+\lambda)&=\cos(45^\circ+\tfrac{1}{2}\alpha)\cos(45^\circ-\tfrac{1}{2}(\epsilon+\delta))\end{aligned}\right\}. \quad (21)$$

As an example of the conversion, showing the most convenient arrangement of the work, let us convert the equatorial coordinates of α Lyrae for 1900 into longitude and latitude. The data are

		h.	m.	s.
R.A. of α Lyrae, - -	$\alpha=$	18	33	33·162
	$=$	278°	23′	17″·43
Dec. „ „ - -	$\delta=$	38	41	25 ·71
Obliquity of the ecliptic,	$\epsilon=$	23	27	8 ·26

Usual Method.

$\sin\alpha$	9·9953291 n		
$\cos\delta$	9·8923920		
$\cos\alpha$	9·1639923 n		
$\sin\delta=m\sin N$	9·7959584		
$m\cos N$	9·8877211 n		
$\tan N$	9·9082373 n		
N	141°	0′	30″·65
ϵ	23	27	8 ·26
$N-\epsilon$	117	33	22 ·39
$\sin(N-\epsilon)$	9·9477069		
$\log m$	9·9971663		
$\cos(N-\epsilon)$	9·6652232 n		
$\sin\beta$	9·9448732		
$\cos\beta\sin\lambda$	9·6623895 n		
$\cos\beta\cos\lambda$	9·0563843 n		
$\tan\lambda$	0·6060052		
$\cos\beta$	9·6753239		
$\tan\beta$	0·2695493		
λ	283°	54′	51″·36
β	61	44	16 ·79

Gaussian Method.

$\frac{1}{2}\alpha$	139°	11′	38″·72
$45°+\frac{1}{2}\alpha$	184	11	38 ·72
$\epsilon+\delta$	62	8	33 ·97
$\epsilon-\delta$	−15	14	17 ·45
$\frac{1}{2}(\epsilon+\delta)$	31	4	16 ·98
$\frac{1}{2}(\epsilon-\delta)$	−7	37	8 ·72
$45°-\frac{1}{2}(\epsilon+\delta)$	13	55	43 ·02
$45°-\frac{1}{2}(\epsilon-\delta)$	52	37	8 ·72
$\sin(45°-\frac{1}{2}(\epsilon+\delta))$	9·3814992		
$\cos(45°+\frac{1}{2}\alpha)$	9·9988354n		
$\cos(45°-\frac{1}{2}(\epsilon+\delta))$	9·9870386		
$\cos(45°-\frac{1}{2}(\epsilon-\delta))$	9·7832682		
$\sin(45°+\frac{1}{2}\alpha)$	8·8641271n		
$\sin(45°-\frac{1}{2}(\epsilon-\delta))$	9·9001518		
$\sin(45°-\frac{1}{2}\beta)\sin\frac{1}{2}(E-\lambda)$	9·3803346n		
$\sin(45°-\frac{1}{2}\beta)\cos\frac{1}{2}(E-\lambda)$	8·6473953n		
$\tan\frac{1}{2}(E-\lambda)$	0·7329393		
$\cos(45°-\frac{1}{2}\beta)\sin\frac{1}{2}(E+\lambda)$	8·7642849n		
$\cos(45°-\frac{1}{2}\beta)\cos\frac{1}{2}(E+\lambda)$	9·9858740n		
$\tan\frac{1}{2}(E+\lambda)$	8·7784109		
$\frac{1}{2}(E-\lambda)$	259°	31′	17″·04
$\frac{1}{2}(E+\lambda)$	183	26	8 ·44
$\sin(45°-\frac{1}{2}\beta)$	9·3876385		
$\cos(45°-\frac{1}{2}\beta)$	9·9866552		
$\tan(45°-\frac{1}{2}\beta)$	9·4009833		
$45°-\frac{1}{2}\beta$	14°	7	51″·62
$\frac{1}{2}\beta$	30	52	8 ·38
β	61	44	16 ·76
λ	283	54	51 ·40
E	82	57	25 ·48

The difference between the results of two computations $\Delta\lambda = 0''\cdot04$ and $\Delta\beta = 0''\cdot03$, arises from the imperfections of the logarithms, due to the neglect of decimals after the seventh.

As to length of computation, although there are more lines of numbers to be written when the Gaussian equations are used, the numbers of entries of logarithmic tables is about the same in the two methods.

53. Check computations.

It is desirable that the accuracy of every computation be tested. As tests of the above transformations we have

$$\left.\begin{aligned} &\cos M \cos\delta \sin\alpha = \cos(M+\epsilon)\cos\beta\sin\lambda \\ \text{and}\quad &\cos N\cos\beta\sin\lambda = \cos(N-\epsilon)\cos\delta\sin\alpha \end{aligned}\right\} \quad\ldots\ldots\ldots(22)$$

The following more complete test is that of Tietjen.* It consists in computing the differences, generally not large, $\lambda-\alpha$ and $\delta-\beta$, independently from the final results, and comparing them with those found by subtraction.

TIETJEN'S TEST EQUATIONS.

$$\left.\begin{aligned} \sin(\lambda-\alpha) &= 2\cos\alpha\sec\beta m \sin\tfrac{1}{2}\epsilon\sin(M+\tfrac{1}{2}\epsilon) \\ &= 2\cos\alpha\sec\beta m\sin\tfrac{1}{2}\epsilon\sin(N-\tfrac{1}{2}\epsilon) \end{aligned}\right\} \quad\ldots\ldots(23)$$

$$\left.\begin{aligned} \sin\tfrac{1}{2}(\delta-\beta) &= \sec\tfrac{1}{2}(\delta+\beta) m\sin\tfrac{1}{2}\epsilon\cos(M+\tfrac{1}{2}\epsilon) \\ &= \sec\tfrac{1}{2}(\delta+\beta)m\sin\tfrac{1}{2}\epsilon\cos(N-\tfrac{1}{2}\epsilon) \end{aligned}\right\} \quad\ldots\ldots(24)$$

The first equation becomes doubtful as a test for large values of β, because $\sec\beta$ is then large. The following similar ones, derived by applying Napier's analogies to the parts of the triangle *EPS*, seem to be a little shorter in computation, and less liable to the above-mentioned drawback.

$$\left.\begin{aligned} \sin\tfrac{1}{2}(\lambda-\alpha) &= \tan\tfrac{1}{2}\epsilon\cos\tfrac{1}{2}(\lambda+\alpha)\tan\tfrac{1}{2}(\delta+\beta) \\ \tan\tfrac{1}{2}(\delta-\beta) &= \tan\tfrac{1}{2}\epsilon\sin\tfrac{1}{2}(\lambda+\alpha)\sec\tfrac{1}{2}(\lambda-\alpha) \end{aligned}\right\} \quad\ldots\ldots(25)$$

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* *Berliner Jahrbuch*, 1879. OPPOLZER, *Bahnbestimmung*, 1, 13.

The following is the complete computation of the last test:

λ	283°	54′	51″·40
α	278	23	17 ·43
δ	38	41	25 ·71
β	61	44	16 ·76
$\delta+\beta$	100	25	42 ·47
$\lambda+\alpha$	562	18	8 ·83
$\lambda-\alpha$	5	31	34 ·0
$\frac{1}{2}\epsilon$	11	43	34 ·13
$\frac{1}{2}(\delta+\beta)$	50	12	51 ·24
$\frac{1}{2}(\lambda+\alpha)$	281	9	4 ·42
$\frac{1}{2}(\lambda-\alpha)$	2	45	46 ·98
$\tan\frac{1}{2}(\delta+\beta)$	0·0794865		
$\cos\frac{1}{2}(\lambda+\alpha)$	9·2864548		
$\tan\frac{1}{2}\epsilon$	9·3171562		
$\sin\frac{1}{2}(\lambda+\alpha)$	9·9917221n		
$\sec\frac{1}{2}(\lambda-\alpha)$	·0005052		
$\sin\frac{1}{2}(\lambda-\alpha)$	8·6830975		
$\tan\frac{1}{2}(\delta-\beta)$	9·3093835n		
$\frac{1}{2}(\lambda-\alpha)$	2°	45′	46″·97
$\frac{1}{2}(\delta-\beta)$	−11	31	25 ·54
$\lambda-\alpha$	5	31	33 ·94 (97)
$\delta-\beta$	−23	2	51 ·08 (05)
λ	283	54	51 ·37
β	61	44	16 ·79

It will be seen that the test values of λ and β agree better with the results of the usual method than with those of the Gaussian equations.

54. Effect of small changes in the coordinates.

Supplementary to this problem we have that of finding the effect of small changes in the values of one pair of coordinates upon the values of the other, and of converting proper motions or differential variations. For this purpose we require the angle at S of the spherical triangle $PC'S$, which is given by either of the equations

$$\sin S = \cos\lambda \sec\delta \sin\epsilon = \cos\alpha \sec\beta \sin\epsilon, \quad \text{.........(26)}$$

S being taken between the limits $-90°$ and $+90°$.

The required differential coefficients may be found by putting the relations between the parts of the spherical triangle EPS into one of the forms (cf. § 6)

$$\alpha, \delta = f(\lambda, \beta, \epsilon)$$

or

$$\lambda, \beta = f(\alpha, \delta, \epsilon),$$

and may be derived from the differential relations given in Appendix I, on the system explained in § (6). Referring to Fig. 8 and the conventional notation of the sides of the triangle PES in (11), we see that these forms require the relations among the following combinations of parts of the triangle:

For α; parts b, c, A, B.

For δ; parts a, b, c, A.

For λ; parts a, c, A, B.

For β; parts a, b, c, B.

The relations between the differentials of the parts which enter into these four combinations are respectively

$$-\sin C db + \cos a \sin B dc + \sin b \cos C dA + \sin a dB = 0,$$
$$-da + \cos C db + \cos B dc + \sin c \sin B dA = 0,$$
$$-\sin C da + \cos b \sin A dc + \sin b dA + \sin a \cos C dB = 0,$$
$$\cos C da - db + \cos A dc + \sin a \sin C dB = 0.$$

In these general relations we substitute the expressions for the parts and their differentials in terms of α, λ, etc., as formed from (11). We thus find,

$$\left.\begin{aligned} \cos\delta d\alpha &= \cos S \cos\beta d\lambda - \sin S d\beta - \sin\delta \cos\alpha d\epsilon \\ d\delta &= \sin S \cos\beta d\lambda + \cos S d\beta + \sin\alpha d\epsilon \end{aligned}\right\}. \quad (27)$$

and, conversely,

$$\left.\begin{aligned} \cos\beta d\lambda &= \cos S \cos\delta d\alpha + \sin S d\delta + \sin\beta \cos\lambda d\epsilon \\ d\beta &= -\sin S \cos\delta d\alpha + \cos S d\delta - \sin\lambda d\epsilon \end{aligned}\right\}. \quad (28)$$

In using these equations it is usual, following Gauss, to use E, the complement of S, instead of the latter. To use E we have only to write $\cos E$ for $\sin S$ and to take E between the limits 0° and 180°.

55. A more luminous view of the problem will be obtained by geometric construction. Consider Fig. 9 to represent an infinitesimal region around the star infinitely magnified. Let S be the original position of the star. Consider the effect of an infinitesimal displacement SS' upon its coordinates.

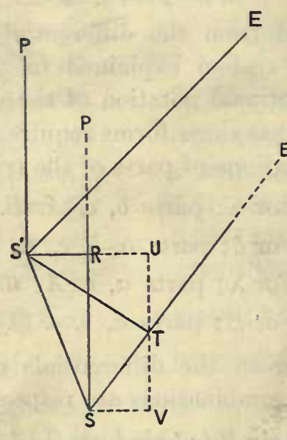

FIG. 9.

Let SP and $S'P$ be arcs of the meridian and SE, $S'E$ arcs from the pole of the ecliptic to the star. Draw $S'T$ and $S'R$ perpendicular to SE and SP, and TV parallel to SP. We then have

$$\left.\begin{aligned} S'R &= \cos\delta\Delta\alpha \\ SR &= \Delta\delta \\ S'T &= \cos\beta\Delta\lambda \\ ST &= \Delta\beta \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a)$$

The transformation of $\Delta\lambda$ and $\Delta\beta$ into $\Delta\alpha$ and $\Delta\delta$ is homologous with the transformation of rectangular coordinates from a system in which SE is the axis of X to one in which SP is that axis. In fact we have

$$S'R = S'U - SV = S'T\cos S - ST\sin S,$$
$$SR = VT + TU = S'T\sin S + ST\cos S.$$

Comparing with (a),

$$\cos\delta\Delta\alpha = \cos S\cos\beta\Delta\lambda - \sin S\Delta\beta,$$
$$\Delta\delta = \sin S\cos\beta\Delta\lambda + \cos S\Delta\beta,$$

as by the analytic method.

If the logarithms of the results are required, so much of the conversion as does not contain ϵ may be made thus:

$$\left.\begin{aligned} h\cos H &= \cos\beta\Delta\lambda \\ h\sin H &= \Delta\beta \\ \cos\delta\Delta\alpha &= h\cos(S+H) \\ \Delta\delta &= h\sin(S+H) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (29)$$

A similar form is readily constructed for the reverse problem.

56. Problem II. *Given the R.A. and Dec. of two bodies, to find the distance between them and the position angle of the one relative to the other.*

Let S and S' be the bodies and P the pole. The angle PSS' which the great circle joining the two bodies makes with the hour circle through one of them, is then called the *position angle* of S' when referred to S. It is counted from the meridian SP, passing north through S, toward the east. The arc SS' joining the bodies is called their angular distance, and is called the *distance* simply. In the spherical triangle $SS'P$ the angle at P is the difference of the R.A.'s, and PS and PS' are the complements of the given declinations. We use the notation $s = SS'$, the distance of the bodies; p, their position angle. The fundamental theorems of spherical trigonometry then give

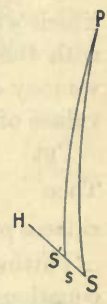

Fig. 10.

$$\left.\begin{aligned} \sin s\sin p &= \sin P\sin PS' = \cos\delta'\sin(\alpha'-\alpha) \\ \sin s\cos p &= \cos\delta\sin\delta' - \sin\delta\cos\delta'\cos(\alpha'-\alpha) \\ \cos s &= \sin\delta\sin\delta' + \cos\delta\cos\delta'\cos(\alpha'-\alpha) \end{aligned}\right\} \quad \ldots\ldots (30)$$

We may transform the last two equations in the usual way for logarithmic computation by computing m and M from the equations

$$m \sin M = \sin \delta',$$
$$m \cos M = \cos \delta' \cos(\alpha' - \alpha).$$

Then

$$\left.\begin{aligned} \sin s \sin p &= \cos \delta' \sin(\alpha' - \alpha) \\ \sin s \cos p &= m \sin(M - \delta) \\ \cos s &= m \cos(M - \delta) \end{aligned}\right\} \quad \text{.............. (31)}$$

will be three equations for computing s and p, with a partial check on the accuracy of the computation. But the check and the third equation will be useless if s is a small arc, say less than 5°.

In the usual applications of this problem $\alpha' - \alpha$ and s are so small that their cosines may be taken as unity. We may then use the equations

$$\left.\begin{aligned} s \sin p &= (\alpha' - \alpha) \cos \delta' \\ s \cos p &= \delta' - \delta \end{aligned}\right\} \quad \text{.................... (32)}$$

It is generally the case that the position angle p is not required with precision, or that, instead of defining this angle as that at P, we may take the mean of the angles PSS' and $PS'H$, which will differ little from the angle which the arc SS' makes with the hour circle through its middle point. In these cases we may derive an approximate formula applicable to yet greater values of s, as follows:

Put p', the exterior angle $PS'H$.

Then

$$\sin s \cos p' = -\sin s \cos PS'S = -\cos \delta' \sin \delta + \sin \delta' \cos \delta \cos(\alpha' - \alpha).$$

Putting for brevity $A = \frac{1}{2}(\alpha' - \alpha)$, the last member of these equations may be written

$$-\cos \delta' \sin \delta (\cos^2 A + \sin^2 A) + \sin \delta' \cos \delta (\cos^2 A - \sin^2 A),$$

whence $\sin s \cos p' = \cos^2 A \sin(\delta' - \delta) - \sin^2 A \sin(\delta' + \delta)$,

while in the same way the second equation (30) gives

$$\sin s \cos p = \cos^2 A \sin(\delta' - \delta) + \sin^2 A \sin(\delta' + \delta).$$

Taking the half sum of these equations and putting $P = \frac{1}{2}(p + p')$, we have

$$\sin s \cos \tfrac{1}{2}(p' - p) \cos P = \cos^2 A \sin(\delta' - \delta). \quad \text{........ (33)}$$

We have also

$$\sin s \sin p' = \cos\delta \sin(\alpha' - \alpha);$$

taking the half sum of this and the first of (30),

$$\sin s \cos\tfrac{1}{2}(p'-p)\sin P = \cos\tfrac{1}{2}(\delta'-\delta)\cos\tfrac{1}{2}(\delta'+\delta)\sin(\alpha'-\alpha). \quad (33a)$$

If s and $\alpha'-\alpha$ are each less than half a degree, we may put $\sin s = s$; $\sin(\alpha'-\alpha) = \alpha'-\alpha$ and $\cos\frac{1}{2}(p'-p) = \cos\frac{1}{2}(\delta'-\delta) = 1$, without serious error. If s and $\alpha'-\alpha$ are less than 15′, the error will generally not exceed 0″01. Thus we shall have from (33) and (33a)

$$\left.\begin{aligned} s\sin P &= (\alpha'-\alpha)\cos\tfrac{1}{2}(\delta'+\delta) \\ s\cos P &= \delta'-\delta \end{aligned}\right\}. \quad \text{(34)}$$

57. To find the effect of small changes in α and δ upon s and p, the last equations are commonly accurate enough, and no distinction will be necessary between P and p so far as the differential values are concerned. By differentiating (34), writing p for P, and putting for brevity $\delta_1 = \frac{1}{2}(\delta'+\delta)$, we find

$$\left.\begin{aligned} \sin p\,ds + s\cos p\,dp &= \cos\delta_1(d\alpha'-d\alpha) - \tfrac{1}{2}(\alpha'-\alpha)\sin\delta_1(d\delta'+d\delta) \\ \cos p\,ds - s\sin p\,dp &= d\delta' - d\delta \end{aligned}\right\}. \quad (35)$$

Eliminating dp by multiplying the first of these equations by $\sin p$ and the second by $\cos p$ and taking their sum, we find

$$\left.\begin{aligned} \frac{ds}{d\alpha'} &= -\frac{ds}{d\alpha} = \sin p\cos\delta_1 \\ \frac{ds}{d\delta'} &= \cos p - \tfrac{1}{2}s\sin^2 p\tan\delta_1 \\ \frac{ds}{d\delta} &= -\cos p - \tfrac{1}{2}s\sin^2 p\tan\delta_1 \end{aligned}\right\}. \quad \text{(36)}$$

Multiplying the first equation (35) by $\cos p$ and the second by $\sin p$ and subtracting, we find

$$\left.\begin{aligned} \frac{s\,dp}{d\alpha'} &= -\frac{s\,dp}{d\alpha} = \cos p\cos\delta_1 \\ \frac{s\,dp}{d\delta'} &= -\sin p(1+\tfrac{1}{2}s\cos p\tan\delta_1) \\ \frac{s\,dp}{d\delta} &= \sin p(1-\tfrac{1}{2}s\cos p\tan\delta_1) \end{aligned}\right\}. \quad \text{(37)}$$

CHAPTER V.

THE MEASURE OF TIME AND RELATED PROBLEMS.

Section I. Solar and Sidereal Time.

58. The main purpose of a measure of time is to define with precision the moment of a phenomenon. The methods of expressing a moment of time fall under two divisions: one relating to what in ordinary language is called the "time of day," and depending on the earth's rotation on its axis; the other on the count of days, which leads us to the use of years or centuries. In any case, the foundation of the system is the earth's rotation. The time of this rotation we are obliged, in all ordinary cases, to treat as invariable, for the reason that its change, if any, is so minute that no means are available for determining it with precision and certainty. There are theoretical reasons for believing that the speed of rotation is slowly diminishing from age to age, and observations of the moon make it probable that there are minute changes from one century to another. If such is the case the retardation is so minute that the change in the length of any one day cannot amount to a thousandth of a second. Yet, by the accumulation of a change even smaller than this through an entire century, the total deviation may rise to a few seconds and, in the course of many centuries, to minutes.

59. Relations of the sidereal and solar day.

In ordinary life the day is determined by the apparent diurnal motion of the sun. The astronomical day, when used for the measure of time, rests on the same basis. The most natural unit of time would be that of one rotation of the earth on its axis. But owing to the annual motion of the earth around the sun, and the consequent continual change of the sun's right ascension, the solar day and the time of the earth's actual rotation are not the same, the latter being nearly four minutes less than the former. Hence, the introduction into astronomical practice of a sidereal day. The *sidereal day*, properly so-called, is the time of the earth's rotation on its axis, and is equal to the interval between two passages of an equatorial star without proper motion over the meridian of a place. The restriction to an equatorial star is necessary because, owing to the continual change in the direction of the earth's axis, known as precession, the actual interval between two culminations of a star varies with its declination.

The sidereal day proper is not used in astronomical practice. Instead of the passage of a star over the meridian, we take the passage of the vernal equinox. The practical sidereal day is the interval between two transits of the equinox over the same meridian. It is divided into 24 sidereal hours, and these into minutes and seconds according to the civil custom. For 0 h. sidereal time, called also *sidereal noon*, is taken the moment of transit of the vernal equinox over the meridian.

Imagine that, at the moment of this transit, we set a clock keeping perfect sidereal time at 0 h. 0 m. 0 s., and compare the apparent motion of the sidereal sphere with the clock. As the hour angle of the vernal equinox continually increases at such a rate that the equinox returns to the meridian in 24 sidereal hours, it increases at the rate of 15° for every hour. It follows that the sidereal clock, when correct, marks at every moment the hour angle of the equinox. Moreover, since the right ascension of a star is equal to the angle between the hour circles through the vernal equinox and through the star, it follows that the clock, at every moment, shows the right ascension of any star which is on the meridian at that moment. In other words, it continually indicates the changing right ascension of the meridian. At the end of 24 sidereal hours the vernal equinox once more reaches the meridian and the clock once more marks 0 h.

It follows from this that, if the moment of culmination of any star of known right ascension is observed, and we set a perfect sidereal clock at that moment so that its face shall indicate the right ascension, the indication of the clock will remain correct through the 24 hours and will show the R.A. of all objects passing the meridian, expressed in units of time. This is, in principle, the way in which right ascensions are determined.

Sidereal time is used in astronomy for the indication of the apparent position of the celestial sphere. As a general measure of time the mean solar day is used.

The natural day is the interval between two culminations of the sun over the meridian. It is divided into hours, minutes, and seconds of solar time. The time determined by starting from the moment of a culmination, and measuring off solar hours, is called *apparent time.* It is equal to the hour angle of the sun at any moment.

Owing to the unequal motion of the sun in right ascension, arising from the obliquity of the ecliptic and the eccentricity of the earth's orbit, the days and hours thus determined are of unequal length, and a clock would have to be continually changed in order to keep apparent time. Hence, this measure of time is entirely out of use for astronomical purposes, and is used in civil life only in regions where uniform time cannot be obtained.

Both the civil and astronomical time now in almost universal use are measured by the transits of a *mean sun* over the meridian. This is a fictitious body moving uniformly along the equator, at such a rate that it shall, in the long run, be as much ahead of the real sun as behind it. The interval between two consecutive transits of this body is called the *mean solar day.* The corresponding time of day is called *mean solar time.* The difference between mean and apparent time is the *equation of time,* which is given in the Ephemeris for every day of the year.

60. Astronomical mean time.

In our common reckoning of time the day begins at midnight, and is divided into two parts of 12 hours each. Time thus expressed is called *civil time.* But in astronomical usage the day begins at noon, and the hours are counted from 0 h. to 24 h., from each noon to the next. Time thus expressed is called *astronomical mean time*; or simply *mean time.*

On this system each day is conceived to continue till noon of the day following, so that, for example, January 2, 9 h. 20 m. A.M., civil time, is, in astronomical time, January 1, 21 h. 20 m. The following precepts for changing civil to astronomical time, and *vice versa*, are obvious:

If the civil time is A.M., *take one from the days, add* 12 *to the hours, and drop* A.M.

If P.M. *drop* P.M.

In either case the result is astronomical time.

To change astronomical to civil time:

If astronomical time is less than 12 *hours write* P.M. *after it.*

If greater, subtract 12 *hours, add* 1 *to the days, and write* A.M.

61. Time, longitude, and hour angle.

Since the hour angle of the mean sun increases by 360° in a mean solar day, it follows that it increases by

$$\left.\begin{array}{l} 15^\circ \text{ in 1 hour} \\ 15' \text{ in 1 minute} \\ 15'' \text{ in 1 second} \end{array}\right\} \text{of mean solar time.}$$

For a similar reason, the hour angle of the vernal equinox increases at the rate of

$$\left.\begin{array}{l} 15^\circ \text{ in 1 hour} \\ 15' \text{ in 1 minute} \\ 15'' \text{ in 1 second} \end{array}\right\} \text{of sidereal time.}$$

Moreover, as the earth rotates, mean noon passes over 15° of longitude in 1 hour of mean time, and sidereal noon in 1 hour of sidereal time. Thus we may say, in a general way, that time, expressed in hours, minutes, and seconds, may be changed into arc (°, ′, ″) by multiplying by 15. To save this multiplication it is common to express right ascension, hour angle, and terrestrial longitude in time. This is equivalent to dividing the circle into 24 hours instead of 360°, so that 6 hours make a quadrant. There will then be 4 m. in every degree and 4 s. in every minute of arc.

62. Absolute and local time.

Since noon, or any other hour of the day, travels continuously round the world, it follows that the moment when any day or year begins or ends varies with the longitude of the place. According to the custom now generally prevalent, noon of any day, say January 1, begins when the sun crosses the 180th meridian from Greenwich, and ends when the sun gets back to that meridian. Hence local time at a common moment may differ by any amount less than 24 hours for two places on opposite sides of the 180th meridian. We must therefore distinguish between

Absolute time, which is any common measure of time to be used for all places, and

Local time, which depends on the longitude of the place.

The daily affairs of life are controlled by local time, which is also the only time that can be readily and directly determined by astronomical observations. If we have to compare moments noted at different places we must reduce each moment to some common standard of time which we regard as absolute.

The usual standard of absolute time is the local time of some prime meridian, generally that of Greenwich. But we may equally use time defined without reference to any meridian; for example, time counted from the moment when the sun crossed the vernal equinox. Such a system was proposed by Sir John Herschel, but has never come into use, because it is less convenient than Greenwich time.

Local time is reduced to Greenwich time by adding the longitude of the place when West; subtracting it when East; Greenwich time is reduced to local time by the reverse operation.

Astronomical custom is divided as to whether East or West longitudes shall be considered positive; the West are positive in the *American Ephemeris*. To avoid ambiguity it is better to use the signs E or W, except where + or − is necessary. In this case we use the notation:

λ = the West longitude of a place from Greenwich expressed in time;

t', the local time;

t, the Greenwich time; then when t' is given

$$t = t' + \lambda,$$

and when t is given, $$t' = t - \lambda.$$

Moments of local time in widely separated parts of the world may be compared by reducing each to Greenwich time.

63. Recapitulation and illustration.

The following is a recapitulation and statement of the fundamental definitions and propositions relating to the subject of time.

I. The mean sun is a fictitious body, increasing uniformly in right ascension at the rate of 24 hours, or 360°, in a solar year, and so placed that the true sun shall on the average be as much behind it as ahead of it.

II. Mean noon at any place is the moment when the mean sun crosses the meridian of that place.

III. Mean time at any place and at any moment is the West hour angle of the mean sun at that place and moment, each 15° of arc counting one hour. It is zero at noon, and may be expressed in hours, minutes, and seconds, or in fractions of a day, from one noon to the next.

IV. Sidereal noon, or sidereal 0 h. at any place, is the moment when the vernal equinox crosses the meridian of that place.

V. Sidereal time at any moment is the West hour-angle of the vernal equinox, and is identical with the right ascension of the meridian at that moment.

VI. Hence the sidereal time at which any object crosses the meridian is its right ascension at the moment of crossing.

VII. The difference between mean and sidereal time at any moment, being the difference between the hour angles of the mean sun and the vernal equinox at that moment, is the right ascension of the mean sun in the sense

$$\text{Sid. Time} - \text{Mean Time} = \text{R.A. Mean Sun}.$$

VIII. Hence, at any one and the same moment of absolute time the difference between sidereal and mean time is the same at all places on the earth's surface.

IX. Hence, also, the sidereal time of mean noon is identical with the right ascension of the mean sun at that time.

Illustration of the propositions. Let O be the centre of the earth, and the plane of the paper that of the equator, seen from the north side. On this plane project

G, the position of Greenwich;
P, the position of any other place;
OS, the direction of the mean sun;
OE the direction of the vernal equinox.

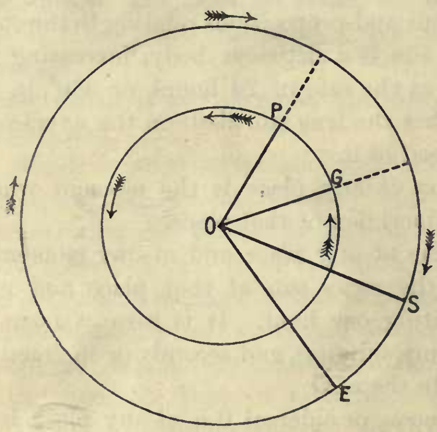

FIG. 11.

The inner arrows show the real direction of rotation of the earth; the outer ones the apparent direction of the rotation of the celestial sphere. We then have:

Angle GOP, the East longitude of P;
" GOS, the Greenwich mean time;
" POS, the local mean time of P,
" GOE, the Greenwich sidereal time,
" POE, the local sidereal time of P
" SOE, the R.A. of the mean sun.

64. Effect of nutation.

Since the equinoxial point does not move uniformly along the equator, we introduce a fictitious point called the *mean equinox*, which moves uniformly, a minute gradual increase from century to century excepted. The difference in R.A. between the mean and true equinoxes is called *Nutation in Right Ascension*, and is given for every tenth day in the Astronomical Ephemeris. Its greatest amount is about 18″ or 1·20 s.

The R.A. of the mean sun is measured from the actual equinox. But its motion can be uniform only when measured from the mean equinox. Sidereal time is measured by the transits of the actual equinox, affected by nutation. Hence its units are not perfectly invariable. But since the irregularity does not amount to more than a fraction of a second in a year, it is entirely insensible from day to day. Sidereal time being not used as a measure of time through long periods, this irregularity causes no inconvenience.

65. The year and the conversion of mean into sidereal time, and vice versa.

The solar year is the interval between two passages of the mean sun through the mean vernal equinox. Its length is

$$\text{Solar Year} = 365{\cdot}24220 \text{ days.}$$

It is evident that since the sun and the equinox are again together at the end of the year, the equinox has made one apparent diurnal revolution more than the sun. Hence

$$365{\cdot}2422 \text{ solar days} = 366{\cdot}2422 \text{ sidereal days.}$$

The ratio of these two numbers is a factor by which intervals of solar time may be changed to sidereal time or *vice versa*. The most convenient form for using the factors in question is reached by putting

$$k = \frac{1}{365{\cdot}242}; \quad k' = \frac{1}{366{\cdot}242}.$$

Then

$$\text{Sid. Time} = \text{M.S.T.} \times (1 + k),$$

$$\text{M.S.T.} = \text{Sid. Time} \times (1 - k').$$

In the *American Ephemeris* (Appendix) and in most collections of astronomical tables, and in Appendix II. of the present work, tables of products of intervals of time by k and k' are given. They are based on the equations

$$24 \text{ h. solar time} = (24 \text{ h.} + 3 \text{ m. } 56{\cdot}556 \text{ s.}) \text{ sid. time},$$
$$24 \text{ h. sid. time} = (24 \text{ h.} - 3 \text{ m. } 55{\cdot}910 \text{ s.}) \text{ solar time}.$$

The reduction may be made by taking the proportional parts of these corrections for the given interval.

If tables are not at hand and the conversion is not required to a higher degree of precision than 0·1 s., a sidereal interval may be reduced to a solar one by the following rule:

Divide the given sidereal interval by 6, taking the seconds as reduced to decimals of a minute, and write the hours of the quotient in the minute column, and the minutes in the seconds column. Diminish the quotient by $\frac{1}{60}$ of its amount; the remainder will be the reduction.

As an example reduce 13 h. 4 m. 17·8 s. sidereal time to solar time:

	h.	m.	s.	
6)	13	4	17·8	
60)		−2	10·72	
		+	2·18	
	13	2	9·3	= interval of M.S.T.

For the reverse reduction, divide the first quotient by 70 instead of 60. As an example

	h.	m.	s.	
6)	13	2	9·3	m. solar interval.
70)		+2	10·36	
		−	1·86	
	13	4	17·8	sidereal interval.

In each example we have added hundredths of a second to avoid an accumulation of errors in the tenths.

The preceding conversion is only that of intervals between two moments of the same day. To convert an actual time of day, we must know the sidereal time of mean noon of the day in question. This requires us to consider the general method of measuring and expressing time through all the centuries.

Section II. The General Measure of Time.

66. In Astronomy time is commonly treated as a continually flowing quantity, which it really is. But in common life certain portions, as days, months, or years, are counted as if they were separate pieces distinguished by ordinal numbers. For example, that year which began with the assumed moment of the birth of Christ is called the first year, or the year 1, and in common language, any event which happened during that year, were it only the day after Christ, would be said to have happened in the year 1.

But, if we consider time as continually flowing, and express the interval from Christ's birth until any moment in years and decimals, then for any moment during the first year the interval would be only a fraction of a year; for example, on April 1, it would be 0·25 y.; or 0 year, 3 months. Carrying forward the count through nineteen centuries we see that April 1, 1900, was really only 1899·25 years from the beginning of our era. In general, when time is measured continuously the integral number of years is less by 1 than when each of its units is taken as an ordinal number.

To avoid the inconvenience thus arising astronomers measure the years from a zero epoch one year earlier than the birth of Christ; that is, they place a year 0 before the year 1, and measure from its beginning. Thus, a moment at the middle of the year 1900 would be designated 1900·5, although only 1899·5 years would have elapsed since the Christian era.

This system leads to a difference of one year between the astronomical notation and that of chronologists in designating dates B.C. The two systems are shown graphically as follows, the horizontal line representing the course of time from left to right, and the vertical lines marking the beginnings of the years. Above the line are the numbers assigned to the years by the notation of the astronomers; below it those of the civil time of the chronologists.

Astronomical	−1 y.	0 y.	1 y.	2 y.	3 y.
Civil - -	B.C. 2	B.C. 1	A.D. 1	A.D. 2	A.D. 3

The same system is extended to the days of the year and month. Mean noon of January 1 is called January 1·0. The zero epoch from which this one day is measured is noon of December 31. Hence the commencement of the astronomical year may be said to be noon of December 31, which is often called January 0. We may regard December 31 as belonging to either year. Thus the moment of 6 o'clock P.M., on 1899 December 31, may be called

either 1899, December 31·25,
or 1900, January 0·25;

while 6 o'clock A.M. of 1900, January 1, may be called

either 1899, December 31·75,
or 1900, January 0·75.

67. Units of time: the day and year.

The fundamental unit for measuring long intervals of time, when the greatest precision is required, is the mean solar day, as already defined. Taking any fixed date as a fundamental epoch, we may express any moment in history by the number of days and the fraction of a day before or after this epoch. One system of doing this, which has the advantage of being continuous through all history, is that of using days of the Julian period. The latter is taken to begin 4713 years before the Christian era, and, in our time,

1900, January 0 = 2415020 days of the Julian period.

As in all our records time is expressed in years, there is an inconvenience in using days alone in computation. Hence the year is also used as an astronomical unit of time, and that of two kinds, the *Julian* and the *solar*.

The Julian year of $365\frac{1}{4}$ days is used when great precision is required. The number of Julian years and solar days from any date is easily found, due allowance being made for the change from the Julian to the Gregorian Calendar, and for the fourths of a day which enter into the result.

68. The solar or Besselian year.

The solar year is used in computations relating to the fixed stars. It is introduced and based on the following data: At the fundamental epoch 1900, January 0, Greenwich mean noon, the R.A. of the fictitious mean sun, referred to the mean equinox, and affected by aberration was

$$18 \text{ h. } 38 \text{ m. } 45{\cdot}836 \text{ s.},$$

and its motion in a Julian year is

$$24 \text{ h. } 0 \text{ m. } 1{\cdot}84542 \text{ s.},$$

with a minute acceleration through several centuries, arising from a slight acceleration of the precession of the equinoxes.

Putting

τ, the R.A. of the mean sun at any time;

T, the time after 1900, January 0, Greenwich M. Noon,

reckoned in Julian centuries of 36 525 days; we have

$$\tau = 18 \text{ h. } 38 \text{ m. } 45{\cdot}836 \text{ s.} + 8\,640\,184{\cdot}542 \text{ s. } T + 0{\cdot}0929 \text{ s. } T^2.$$

In astronomical practice we take for the beginning of a solar year the moment when

$$\tau = 280° = 18 \text{ h. } 40 \text{ m.},$$

this falling as nearly as may be to the beginning of the Gregorian civil year. It will be seen from the expression for τ that the beginning of the solar year 1900 occurred after the fundamental epoch January 0 by the interval necessary for the mean sun to move through the arc

$$18 \text{ h. } 40 \text{ m.} - 18 \text{ h. } 38 \text{ m. } 45{\cdot}836 \text{ s.} = 74{\cdot}164 \text{ s.}$$

This interval in decimals of a day is

$$\frac{74{\cdot}164 \times 365{\cdot}25}{86402} = 0{\cdot}313\,52 \text{ day},$$

so that the solar year 1900 began at 1900, January 0·313 52, Greenwich M.T., which is 1900, January 0·0995, Washington M.T. This, it will be noted, is a moment of absolute time, having no reference to any special meridian. The solar year thus defined is sometimes called the *Besselian fictitious year*, after Bessel, who first introduced it into astronomy.

The beginnings of preceding or subsequent years may be found by continual addition or subtraction of 365·2422 d. Thus is formed the table for the present century found in Appendix II.

69. We now return to the relation between solar and sidereal time. The fundamental quantity on which this relation depends is the sidereal time of mean noon of any date at any longitude. This is the same as the right ascension of the mean sun at the moment of mean noon on that longitude. As mean noon travels continuously round the earth, it follows that the sidereal time, or the mean right ascension in question, increases continuously at the rate of 3 m. 56·556 s. for each mean solar day, that is for each apparent revolution of the mean sun. We have also seen that the value of the quantity in question for the fundamental Greenwich noon on 1900, January 0, is 18 h. 38 m. 45·836 s. This would be the sidereal time of mean noon for this meridian and this date, when referred to the mean equinox. But, in astronomical practice, as we have already remarked, the equinox taken for reference is the true equinox of the date, which may vary by a little more than 1 s. from the mean equinox. It is, therefore, necessary to add the nutation in right ascension, in order to obtain the sidereal time of noon. As the latter is given in the ephemerides, its computation is not necessary except for epochs for which no ephemerides are available.

Section III. Problems Involving Time.

70. Problems of the conversion of time.

In this section we use the abbreviation

S.T.M.N. = Sidereal Time of Mean Noon.

The ordinary problems of conversion of time are the first three following:

PROBLEM I. *From the Greenwich S.T.M.N. to find that of the corresponding date on any other meridian of West longitude* λ.

Since mean noon requires the mean time λ to move over longitude λ, the G.S.T. required for the motion will be λ changed to sidereal time. But the local S.T. will be less than the Greenwich S.T. by λ. Hence the local S.T.M.N. will be greater than the G.S.T.M.N. by the reduction of λ from mean to sidereal time, or

$$\text{S.T.M.N.} = \text{G.S.T.M.N.} + k\lambda.$$

The quantity $k\lambda$ may be taken from any table for the conversion of mean time into sidereal time. In the precept λ must be taken positively toward the West.

PROBLEM II. *To convert mean time into sidereal time.*

The study of the following examples will render a rule unnecessary:

Convert 1905, Jan. 4, 8 h. 49 m. 26·36 s. M.T. of Mt. Hamilton, Cal. (Long. = 8 h. 6 m. 35 s. W.) into sidereal time.

The S.T. of the given moment is equal to the S.T.M.N. plus the interval since M.N. (M.T.) reduced to sidereal time. The first of these quantities is G.S.T.M.N. + $k\lambda$ (Prob. I.) and the second is M.T. $(1+k)$. We take from the ephemeris

	h.	m.	s.
Greenwich S.T.M.N. 1905, Jan. 4 -	18	53	41·85
Reduction to Mt. Hamilton $k\lambda$ (Prob. I.)		1	19·93
Mt. Hamilton S.T.M.N. - - - -	18	55	1·78
Mt. Hamilton mean time, as given -	8	49	26·36
Reduction to sidereal time - - - -		1	26·97
Mt. Hamilton Sidereal Time - -	3	45	55·11

Another method of solution, which is sometimes more convenient, especially when only an approximate result is wanted, makes use of the mean time of sidereal 0 h., found on P. III. of each month of the Ephemeris. The subtractive reduction of this M.T. of Sid. 0 h. to any longitude is found by reducing the West longitude from sidereal to mean time. Thus the above example may be worked as follows:

	h.	m.	s.
Red. of λ = 8 h. 6 m. 35 s. to M.T. - -	0	1	19·72
G.M.T. of Sid. 0 h. (Ephemeris), Jan. 4 -	5	5	27·97
Mount Hamilton M.T.S. 0 h. - - -	5	4	8·25
Given mean time, Jan. 4 - - - -	8	49	26·36
Interval in mean time - - - - -	3	45	18·11
Red. to sidereal time - - - - - -	0	00	37·01
Sidereal time - - - - - - -	3	45	55·12

If the given moment of mean time is before sidereal 0 h. of the same days, the sidereal 0 h. of the day preceding should be used.

PROBLEM III. *To convert sidereal time into mean time.*

Subtract from the S.T. the S.T.M.N., and we have the sidereal interval since mean noon. Convert this into mean time, and the result will be the corresponding mean time.

Reversing the example of Problem II. we have:

	h.	m.	s.
Given sidereal time - - - -	3	45	55·11
S.T.M.N. - - - - - -	18	55	1·78
Sidereal interval since noon - -	8	50	53·33
Reduction to solar time - -		– 1	26·97
Mean solar time - - - - -	8	49	26·36

71. Related problems.

PROBLEM IV. *The right ascension of a body being given, to find its hour-angle at a given moment of mean time, and vice versa.*

From definitions already given it follows that the hour-angle of a body is the difference between its right ascension and that of the meridian. But the latter is equal to the sidereal time. Hence, putting

$$h = \text{the West hour-angle},$$

we have
$$h = t - \alpha,$$

h being taken positively toward the west. Hence the rule:

Convert the given mean time into sidereal time, and from the latter subtract the R.A. The remainder is the hour-angle: West when positive; East when negative.

In the converse problem the hour-angle is given, and the mean time is required.

Since
$$t = h + \alpha,$$

we have the rule:

To the R.A. add the hour-angle. The sum is the sidereal time, which may be converted into mean time.

COR. To find the moment at which a heavenly body of known R.A. crosses the meridian, we have only to take its R.A. as sidereal time, and convert it into mean time.

PROBLEM V. *To find the mean time at which the moon culminates at a given place, on a given day, and its R.A. and Dec. at the moment of culmination.*

This problem cannot be solved so simply as that preceding because the R.A. of the moon is continually changing, and is therefore not a given quantity. What is given in the *Ephemeris* is the moon's R.A. for every hour of G.M.T. This R.A. never changes by more than an hour in any one day. Hence if we take the nearest hour of R.A. for the middle of the day and add to it mentally the M.T.S.N. and the West longitude, so as to get the sum to the nearest round hour, this sum will be the G.M.T. of culmination at the local meridian within at least one or two hours. By repeating the process, using minutes, we shall have the G.M.T. within 5 minutes, and can thus find the nearest hour of G.M.T. mentally.

Of course the hour selected need not be the absolutely nearest one. Near the half hour, either the hour preceding or following, may be taken.

For this selected hour of G.M.T. take out or compute

α_0, the moon's R.A.;

α', the change of R.A. for 1 m. of mean time;

τ_0, the local sidereal time.

Were the selected hour exactly that of culmination, we should have

$$\tau_0 = \alpha_0.$$

But as this equation will not be satisfied, we must find a number t of minutes before or after the hour at which the equation

$$\tau = \alpha$$

is satisfied, τ being the local sidereal time. Now, in one minute of mean time τ changes by

$$60 \text{ s. } (1 + K) = 60{\cdot}1643 \text{ s.}$$

Hence, at t minutes after the hour,

$$\tau = \tau_0 + 60{\cdot}1643 \text{ s} \times t,$$
$$\alpha = \alpha_0 + '\alpha\, t.$$

Equating these values, we have

$$t = \frac{(\alpha_0 - \tau_0) \text{ (in seconds)}}{60{\cdot}1643 \text{ s.} - \alpha'},$$

and

$$\alpha = \alpha_0 + t\alpha' = (\alpha_0 - \tau_0)\frac{\alpha'}{60{\cdot}1643 \text{ s.} - \alpha'} + \alpha_0.$$

The declination may then also be interpolated to the time t by the formula
$$\delta = \delta_0 + t\delta'.$$

When many culminations are to be computed the factor

$$\frac{\alpha'}{60{\cdot}1643 \text{ s.} - \alpha''}$$

or its logarithm, may be tabulated for every 0·01 s. of α'.

EXAMPLE. *To find the time of culmination of the moon on 1907, June 6, at San Francisco,* $\lambda = 8$ h. 10 m. West.

Looking at pp. 94 and 97 of the *Ephemeris*, we find by using the second method of converting the Moon's R.A. as S.T. into M.T.

19 h. + 2 h. + 8 h. = 29 h. or 5 h. G.M.T.

Thus the first approximation is

5 h. G.M.T. or 21 h. local M.T.

Now, 21 h. M.T. is 9 o'clock A.M. of the civil day next following, and if we wish the culmination on the morning of June 6, civil time at San Francisco, we must take as the starting point of computation

June 5, 21 h. local M.T. = June 6, 5 h. G.M.T.

Then, our second approximation will be

	h.	m.
M.T.S.N., *Eph.* p. 94 - - -	19	1
Moon's RA., *Eph.* p. 97 - - -	1	47
Longitude of place - - -	8	10
	4	58

So 5 h. G.M.T. is really the nearest hour. With this time as argument, we take from the *Ephemeris*

$$\alpha_0 = 1\text{ h. }46\text{ m. }49{\cdot}69\text{ s.} \qquad \alpha' = +1{\cdot}9502\text{ s.}$$

The local sidereal time must be accurately computed. It is found to be

$$\tau_0 = 1\text{ h. }45\text{ m. }48{\cdot}81\text{ s.}$$

We now have all the data for the computation of t and α. The preceding formulae give

$$t = +1{\cdot}0458\text{ m.} = +1\text{ m. }2{\cdot}75\text{ s.},$$

and hence the increment to be added to α_0 is

$$\Delta\alpha = '\alpha t = +2{\cdot}04\text{ s.}$$

The required time of culmination is therefore

1907, June 6, 8 h. 51 m. 2·75 s. A.M.

The right ascension of the moon at this time is

$$\alpha = 1\text{ h. }46\text{ m. }51{\cdot}73\text{ s.}$$

In some cases, another approximation will be found necessary, if the greatest accuracy is desired. On account of the time falling so close to an even hour, in the above problem, such further approximation is not necessary, the result obtained being accurate to the nearest hundredth of a second.

COR. To find the time when the moon has a given geocentric hour-angle h at a given place, we find the time of its culmination over a meridian whose longitude is h west of the given place, or $h+\lambda$ west of Greenwich.

PROBLEM VI. *The R.A. and Dec. of a star being given to find its altitude, azimuth, and parallactic angle at a given time.*

Let $MZPN$ be the meridian.

MN, the horizon.

Z, the zenith.

S, the position of the star.

Then in the spherical triangle PZS,

$PZ =$ co-latitude of place $= 90° - \phi$.

$PS =$ N.P.D. of star $= 90° - \delta$.

$SH =$ altitude of star $\equiv a$.

$ZS =$ zenith dist. of star $= 90° - a \equiv z$.

Angle at $P =$ hour angle $\equiv h$.

Angle at $Z =$ azimuth $\equiv A$.

Angle at $S =$ parallactic angle $\equiv q$.

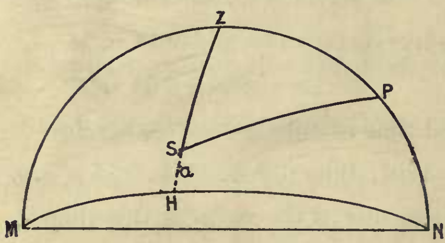

FIG. 12.

The given parts of this triangle are the sides PZ and PS and the included angle P. Hence the Gaussian equations are most convenient when all three of the remaining parts are required. With the above notation the Gaussian equations reduce to

$$\left.\begin{aligned} \sin\tfrac{1}{2}z\sin\tfrac{1}{2}(A-q) &= \cos\tfrac{1}{2}h\sin\tfrac{1}{2}(\phi-\delta) \\ \sin\tfrac{1}{2}z\cos\tfrac{1}{2}(A-q) &= \sin\tfrac{1}{2}h\cos\tfrac{1}{2}(\phi+\delta) \\ \cos\tfrac{1}{2}z\sin\tfrac{1}{2}(A+q) &= \cos\tfrac{1}{2}h\cos\tfrac{1}{2}(\phi-\delta) \\ \cos\tfrac{1}{2}z\cos\tfrac{1}{2}(A+q) &= \sin\tfrac{1}{2}h\sin\tfrac{1}{2}(\phi+\delta) \end{aligned}\right\}\quad\ldots\ldots\ldots(1)$$

The azimuth A thus found will be counted from the North point toward the West, and will be 180° for a point on the meridian south of the zenith.

As a check upon the work we have

$$\frac{\sin z}{\sin h} = \frac{\cos\delta}{\sin A} = \frac{\cos\phi}{\sin q}.$$

The elementary formulae of spherical trigonometry may also be applied to the problem, and will be simpler than the Gaussian formulae, if the parallactic angle and azimuth are not both required. They become, in the present case

$$\left.\begin{aligned} \cos z &= \sin\phi\sin\delta + \cos\phi\cos\delta\cos h \\ \sin z\sin A &= \cos\delta\sin h \\ \sin z\cos A &= \cos\phi\sin\delta - \sin\phi\cos\delta\cos h \\ \sin z\sin q &= \cos\phi\sin h \\ \sin z\cos q &= \sin\phi\cos\delta - \cos\phi\sin\delta\cos h \end{aligned}\right\}\ldots\ldots\ldots\ldots(2)$$

Transforming these equations in the usual way, we have the following formulae for logarithmic computation:

To find z and A.

$$\left.\begin{aligned} k\sin K &= \cos\delta\cos h \\ k\cos K &= \sin\delta \\ \cos z &= k\sin(K+\phi) \\ \sin z\sin A &= \cos\delta\sin h \\ \sin z\cos A &= k\cos(K+\phi) \end{aligned}\right\}\ldots\ldots\ldots\ldots(3)$$

Or eliminating k, we may use the formulae,

$$\tan K = \cot\delta\cos h$$

$$\tan A = \frac{\sin K\tan h}{\cos(K+\phi)},$$

$$\tan z = \frac{\cot(K+\phi)}{\cos A}.$$

To find z and q.

$$\left.\begin{aligned} k'\sin K' &= \cos\phi\cos h \\ k'\cos K' &= \sin\phi \\ \cos z &= k'\sin(K'+\delta) \\ \sin z\sin q &= \cos\phi\sin h \\ \sin z\cos q &= k'\cos(K'+\delta) \end{aligned}\right\}\ldots\ldots\ldots\ldots(4)$$

Or by the briefer formulae,

$$\tan K' = \cot\phi\cos h,$$

$$\tan q = \frac{\sin K'\tan h}{\cos(K'+\delta)},$$

$$\tan z = \frac{\cot(K'+\delta)}{\cos q}.$$

Respecting the briefer formulae it is to be remarked that they may sometimes fail to give as accurate a result as the data admit of, owing to $\tan z$ coming out as the quotient of two small quantities. This will commonly be the case when A or q differs little from 90° or 270°. On the other hand the extended formulae are always accurate. They also afford a partial check upon the accuracy of the computation by the accordance of $\sin z$ with $\cos z$, which the abbreviated formulae do not.

Problem VII. *The altitude or Z.D. of a known body, and the latitude of the place being given, to find the hour-angle and the local time.*

The first of equations (2) gives, for the hour-angle,

$$\cos h = \frac{\cos z - \sin\phi\sin\delta}{\cos\phi\cos\delta} = \sec\phi\sec\delta\sin a - \tan\phi\tan\delta. \quad\ldots(5)$$

The second form will be most convenient when, as in sextant work, we have a number of altitudes of the same body. The value of $\sec\phi\sec\delta$ and of $\tan\phi\tan\delta$ will then be the same for all the altitudes. After finding the product $\sec\phi\sec\delta\sin a$ in natural numbers we subtract $\tan\phi\tan\delta$ from it, and thus have the nat. cosine of h, and can at once find h from a table of natural sines and cosines.

We may transform the first value of $\cos h$ as in spherical trigonometry, thus:

$$\frac{1-\cos h}{1+\cos h} = \tan^2\tfrac{1}{2}h = \frac{\cos(\phi-\delta) - \cos z}{\cos(\phi+\delta)+\cos z}.$$

Putting

$$s = \tfrac{1}{2}(z+\phi+\delta),$$

this equation may be reduced to

$$\tan^2\tfrac{1}{2}h = \frac{\sin(s-\phi)\sin(s-\delta)}{\cos s\cos(s-z)}.$$

Having found the hour-angle h, the sidereal time is given by the equation

$$\tau = \alpha + h,$$

and the mean time is then found by conversion.

This problem is of constant application in navigation, and tables for facilitating its computation are given in treatises on navigation.

PROBLEM VIII. *To find the mean time of sunrise and sunset at a given place.*

The hour-angle at which a body is on the true horizon is called its *semi-diurnal arc.* It is found by putting $z=90°$ in (5), which gives

$$\cos h = -\tan\phi \tan\delta.$$

If in this formula we use the value of δ as given in the ephemeris (the geocentric value), the result will be the geocentric hour-angle at which the body is on the geocentric horizon. This may differ from the geocentric hour-angle when the body is apparently on the sensible horizon owing to the effect of refraction and parallax. Moreover, in the case of the sun and moon, it is the rising and setting of the upper limb and not of the centre which is usually given in almanacs.

Now, when the upper limb of the sun is apparently on the horizon it is really 34′ below it, being elevated by refraction. The centre is 16′ below the limb, or 50′ below the sensible horizon. The parallax may be neglected unless the result is wanted with unusual accuracy. Hence we may put

$$z = 90°\ 50',$$

or

$$\cos z = -0{\cdot}0145;$$

and for the hour-angle,

$$\cos h = -(0{\cdot}0145 \sec\phi \sec\delta + \tan\phi \tan\delta).$$

Since the West hour-angle of the true sun is the apparent time, this equation will give the apparent time of sunset, to which we must apply the equation of time (given in the ephemeris) to obtain the mean time.

For sunrise we subtract h from 12 h. for civil time or from 24 h. for astronomical time, and apply the equation of time as before.

For δ we must of course take the sun's declination not for noon, as given in the ephemeris, but for the time of sunrise and sunset itself. The change in declination during an hour will generally be unimportant, so that we may need only a rough approximation to the time to get the declination.

PROBLEM IX. *To find the time of rising or setting of the moon on a given day at a given place.*

When the moon is on the horizon it is depressed by parallax by a quantity which averages about 57′. By assuming the parallax to have this constant value we shall in our latitudes rarely be led into an error of more than 20 s. The refraction elevates the moon by 34′, and its mean semi-diameter is $15\frac{1}{2}'$. Hence, when the moon's upper limb appears to coincide with the sensible horizon the true geocentric Z.D. of its centre is about $89° \, 52\frac{1}{2}'$, with a range of 4′ on each side of this mean. The formula for the geocentric hour-angle at apparent setting of the upper limb therefore is

$$\cos h = -\tan \phi \tan \delta + 0{\cdot}0022 \sec \phi \sec \delta.$$

But it will generally happen that, owing to the whole disc of the moon not being illuminated, her entire visible portion will disappear before the setting of her upper limb. It is therefore best to take the setting of her centre. For this we shall have

$$\text{geoc. Z.D.} = 89° \, 37',$$

$$\cos h = -\tan \phi \tan \delta + 0{\cdot}0067 \sec \phi \sec \delta.$$

If the risings and settings for a whole year are to be computed for some one place, it will facilitate the work to make a table giving the value of h from this formula for each degree, or each 10′ of δ from $+29°$ to $-29°$.

The first difficulty we meet is that we cannot find the value of δ until we have an approximate time of the phenomenon. The computation of this time, and of the final result, will be facilitated by the "Moon Culminations" of the Ephemeris. Here are given the local mean time of culmination over the Meridian of Washington, the R.A. and Dec. of the moon at the moment of culmination, and the variation of these quantities for one hour of longitude. By this is meant the change in their values while the moon is moving from the meridian of Washington to the meridian 1 h. West of Washington, supposing the motion of the moon to be uniform. For example, if on a certain date we have

Time of Transit over the Meridian of Washington 9 h. 22·6 m.,
Change in 1 hour of Longitude - - - - 2·27 m.,

the local mean time of transit over the meridian 1 hour West of Washington will be, approximately, 9 h. 24·9 m. And the W.M.T. of this transit will be 10 h. 24·9 m.

Moreover, it must be remembered that whenever the geocentric west hour-angle of the moon at a place L is h, then the moon is on the meridian of a place in longitude h west of L. Hence, having found h, let λ be the longitude of L. Then when the moon is setting or rising at L she is on the meridian of a place in longitude $\lambda + h$ or $\lambda - h$ respectively. Let T_1 be the local M.T. of transit over this place. Then

$$T_1 + h$$

will be the local M.T. at L; that is, the time of moon-set at L.

For moon-rise h must be taken negatively.

EXAMPLE. *To find the time of moon-rise and moon-set at San Francisco,* 1892, *June* 1, *the position being*

$$\phi = +37° \ 48',$$
$$\lambda = +3 \text{ h. } 1·4 \text{ m. West of Washington.}$$

From the *Ephemeris* for 1892, p. 388, we find, for this date

	h.	m.
Mean time of transit over Washington	5	58·01
Red. to San Francisco, 3·02 h. × 1·816 m.		5·48
Local M.T. of transit over San Francisco	6	3·49

The rising and setting we seek are those preceding and following this transit. And the first data we require are the declinations of the moon at the time in question.

Let us put τ for the amount by which the semi-diurnal arc differs from 90° or 6 h., *i.e.* $h = 90^c + \tau$, so that $\cos h = -\sin \tau$. In a first rude approximation we may, in the equation

$$\cos h = -\sin \tau = -\tan \phi \tan \delta,$$

put $\sin \tau = \tau$ and $\tan \delta = \delta$. This gives

$$\tau = \tan \phi \times \delta = +0·776 \ \delta.$$

On the date in question we have $\delta = +13°$; $\tau = +10° = +40$ m.; $h = \pm 6$ h. 40 m. roughly. Thus, as a first rude approximation, the moon, at the required moments, was on the meridian

6 h. 40 m. E. of S.Fr.; ∴ 3 h. 39 m. E. of Wash.
6 h. 40 m. W. of S.Fr.; ∴ 9 h. 41 m. W. of Wash.

From the sixth and seventh columns of the Ephemeris we now find the declinations more accurately as follows:

Declination at transit over Washington	+13°	36′
Change for −3·65 h. of longitude -		+47
" " +9·68 h. " - -	−2	6
Declination at rising - - - -	+14	23
Declination at setting - - - -	+11	30

With these values of δ and the known value of ϕ, we now compute the accurate values of the hour angle. The computation is as follows:

	Moon-rise.	Moon-set.
δ	+14° 23′	+11° 30′
tan ϕ	9·8897	9·8897
tan δ	9·4090	9·3085
log (1)	9·2987*n*	9·1982*n*
log 0·0067	7·8261	7·8261
sec ϕ	0·1023	0·1023
sec δ	0·0138	0·0088
log (2)	7·9422	7·9372
subt. log	0·0195	0·0245
cos h	9·2792*n*	9·1737*n*
h	−100° 58′	+98° 35′
=	−6 h. 43·9 m.	+6 h. 34·3 m.

At the moments of moon-rise and moon-set, therefore, the moon was on the meridian of the places whose longitudes referred to Washington are respectively

h.	m.		h.	m.		h.	m.		h.
3	1·4	−	6	43·9	=	−3	42·5	=	−3·708
3	1·4	+	6	34·3	=	+9	35·7	=	+9·595.

The further computation is then as follows:

	h.	m.
Mean Time of transit over Washington -	5	58·0
Change in $-3·708$ h. $(-3·708 \times 1·816$ m.)	-0	6·7
,, ,, $+9·595$ h. $(+9·595 \times 1·816$ m.)	$+0$	17·4
Mean Time of transit over 1st point -	5	51·3
,, ,, ,, ,, 2nd ,, - -	6	15·4

The longitudes of these two points referred to San Francisco are the two values of h found above. We therefore have

Mean Time of moon-rise at San Francisco:

$$1892, \text{ June } 1, 5^{h}\ 51^{m}·3 - 6^{h}\ 43^{m}·9 = \text{May } 31, 23^{h}\ 7^{m}·4$$
$$= \text{June } 1, 11^{h}\ 7^{m}·4 \text{ A.M.}$$

Mean Time of moon-set at San Francisco:

$$1892, \text{ June } 1, 6^{h}\ 15^{m}·4 + 6^{h}\ 34^{m}·3 = \text{June } 1, 12^{h}\ 49^{m}·7$$
$$= \text{June } 2, 0^{h}\ 49^{m}·7 \text{ A.M.}$$

As a test of the sufficiency of the approximation, we now compute the moon's declination, with the times which we have just found as arguments. The result is:

$$\text{At moon-rise, } \delta = +14°23'·6$$
$$\text{At moon-set, } \delta = +11\ 31\ ·3$$

The agreement of these values with the values we started out with, shews that a further approximation is unnecessary.

PROBLEM XI. *To find the sidereal time required for the semi-diameter of the sun or moon to pass the meridian.*

This problem arises when, from the observed R.A. of the sun's or the moon's limb at the moment of transit, it is required to find the R.A. of centre at the moment of transit of the centre.

Let P be the pole, O the centre of the moon, L the point of its limb tangent to PL, $\Delta\alpha$ the angle P, S'' the angular semi-diameter OL expressed in seconds of arc.

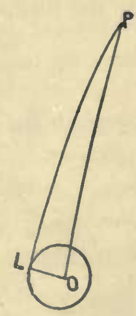

FIG. 13.

Then, (§ 7, Th. ii.) owing to the smallness of the arc s (16′),

$$S'' = \sin PO \,.\, \Delta\alpha = \cos\delta \,.\, \Delta\alpha.$$

Hence, $\Delta\alpha = S'' \sec\delta$,

or when reduced to time, $\Delta\alpha = \frac{1}{15} S'' \sec\delta$.

By this formulae is found the difference of R.A. between the centre and limb at any moment. But what we want is the difference of R.A. at the respective moments of the transits, which is different owing to the change of R.A. during the time occupied by the passage of the semi-diameter. To find the time we put

α, the R.A. of centre at transit.
τ, the sidereal time required for transit of semi-diameter.
α', the change of R.A. in one second of sidereal time.

(=change in 1 m. of mean time $\div$ 60·17. But we may take 60 as the divisor.)

Then; R.A. of centre at transit of limb $= \alpha + \alpha'\tau$.

R.A. of limb „ „ $= \alpha + \alpha'\tau + \Delta\alpha$.

(Because the moment is the same.)

R.A. of limb at transit $= \alpha + \tau$.

Hence, by equating the last two expressions

$$\tau = \frac{\Delta\alpha}{1-\alpha'} = \frac{S'' \sec\delta}{15(1-\alpha')}. \quad \text{.........................(1)}$$

In the case of the planets α' and s are so small that we may use the formula

$$\tau = \frac{S'' \sec\delta}{15}, \quad \text{............................(2)}$$

except in the case of Venus near inferior conjunction.

Remark. The student should be able to shew that (1) will give a correct result for every place by using the geocentric values of S'', δ, and α, instead of their apparent values as seen by the observer.

CHAPTER VI.

PARALLAX AND RELATED SUBJECTS.

Section I. Figure and Dimensions of the Earth.

72. The positions of the heavenly bodies, as found from astronomical tables, and given in ephemerides, are referred to the centre of the earth; while all observations upon them are made on its surface. Hence, in order to express the position of a body referred to the station of an observer, we require a method of reducing its coordinates from the centre of the earth as an origin to any point on its surface. Such a reduction requires a knowledge of the figure and dimensions of the earth. Strictly speaking, what we should know in order to make the reduction with rigour is the actual figure of the earth's surface, including all the inequalities of mountains and valleys, because all points of observations are situated on the actual surface. The figure of the latter being incapable of geometrical definition, the ideal figure used in geodesy is that of the ocean level, and is called the *geoid.* The surface of the geoïd is, at any point, the level to which the water of the ocean would flow if a canal or tunnel were cut from the ocean to the point.

It is a theorem of mechanics that, were the earth homogeneous, the geoid would be an oblate ellipsoid of revolution. In reality, however, the heterogeneity of the earth's interior and the attraction of mountains is such that the surface of the geoid is not rigorously represented by any definable solid. An approximation sufficiently near for most geographical and astronomical purposes is obtained by considering it to be an elliptic spheroid of revolution, affected by small inequalities which are to be determined by observation in each region. In researches relating to parallax the inequalities may, in all ordinary cases, be neglected, and the geoid considered as an ellipsoid of revolution.

Astronomical observations are sometimes made at considerable elevations above the sea level. The Lick Observatory, in California, is at an altitude of 4400 feet. This altitude, at the mean distance of the moon, would subtend an angle of 0″·7; it should, therefore, be taken account of in computing the parallax of the moon. All the other heavenly bodies are so distant that the elevation of the observer above the sea level may be left out of consideration.

73. Local deviations.

The earth's centre being invisible, we have no direct way of determining its direction from any point on the earth's surface. The only line of reference which we can use in the determination of the direction of a heavenly body is the direction of gravity, or that of the plumb line. To refer observations to the centre of the earth, we must ascertain the figure and dimensions of the earth from geodetic measurements on various parts of its surface, combined with observations of the force of gravity, and infer from these where the centre is located.

In doing this, an element of uncertainty is introduced by deviations in the direction of the plumb line due to the non-homogeneity of the earth. Since the attraction varies inversely as the square of the distance, and is exerted by every part of the earth's mass according to the law of gravitation, those portions in the neighbourhood of any region exert a preponderating influence upon the resultant direction of gravity. Hence if the density of the interior is greater on one side of a station than on the opposite side, a deviation will result. In mountainous regions deviations are observed which sometimes amount to 10″, or 20″, or even more. Even in plains far distant from mountains such inequalities amounting to 1″ more or less are the general rule. They are termed *local deviations*. Since the surface of the geoid is everywhere normal to the direction of the plumb line, these deviations shew that it is a spheroid with numerous small inequalities all over its surface.

Since the astronomical observations of altitude are referred to the direction of the plumb line, it follows that there will be corresponding inequalities in the celestial and terrestrial meridians of places on the earth's surface. The plane of the meridian does not, as a general rule, pass rigorously through the axis of the earth. It must be defined as a plane containing the vertical line and parallel to the axis.

This plane defines the apparent celestial meridian of a place in the following way. Imagine that, having determined a north and south line on the earth's surface by the above condition, we follow it a short distance in either direction, and then again determine the meridian. We shall find that the latter will not necessarily be a continuation of the first meridian, but another line making a minute angle with it. In the same way, the celestial meridians of the two points will be great circles intersecting each other at angles which we may regard as infinitesimal. The practical meridian found by starting from any point, and continually travelling in the apparent north and south direction, will be the envelope of the intersections of the system of meridian planes with the earth's surface, and not rigorously the intersection of any one plane. But it is only in refined geodetic work, such as running a meridian line, that the deviation of this envelope from a great circle is of importance. In discussing astronomical observations the local deviation will be unimportant except, possibly, in certain observations of the moon.

74. Geocentric and astronomical latitude.

The inequalities just described, combined with the ellipticity of the earth, lead us to recognize three sorts of terrestrial latitude. One of these—the only one which admits of being determined by direct observation—is the angle between the plumb line and the plane of the equator. As this has to be determined by astronomical observation, it is called the *astronomical latitude.*

The *geocentric latitude* of a point on the earth's surface is the angle which the radius vector drawn from the point to the earth's centre makes with the plane of the equator. This latitude is that which has to be used in computations relating to parallax. It does not admit of direct determination, but has to be determined by correcting the astronomical latitude for the difference between the two latitudes as inferred from geodetic measures generally.

A third latitude, known as *geographic*, is sometimes used. It may be defined as the astronomical latitude corrected for local deviation of the plumb line, or as the angle made with the plane of the equator by a normal to the surface of an imaginary geoid formed by smoothing off the inequalities of the actual geoid so as to reduce it to an ellipsoid of revolution. As this latitude is required only in map-making, where great precision is not necessary, the fact that it does not admit of rigorous determination becomes of little importance.

For our present purpose the problem is to express the coordinates of a point of observation when referred to the centre of the earth as an origin, in terms of the astronomical or geographic latitude.

75. Geocentric coordinates of a station on the earth's surface.

Let Fig. 14 represent a section of the earth through the axis, Y being the north pole. The earth being supposed an ellipsoid

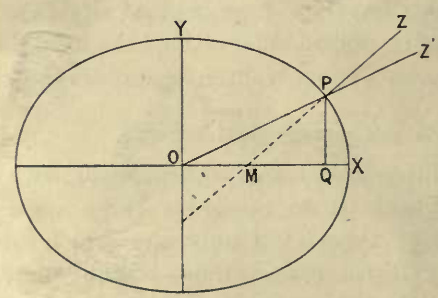

FIG. 14.

of revolution, let P be a point on its surface, PZ the vertical determined by gravity, normal to the surface of the geoid, OPZ'

the radius vector of P from the earth's centre, continued outwards.

Then Z is the apparent or astronomical zenith, Z' the geocentric zenith.

The angle ZPZ' between the apparent and geocentric zeniths is called the *angle of the vertical.*

Omitting local deviation, the geocentric zenith is on the meridian in the direction from the apparent zenith toward the celestial equator. Let us now put:

ϕ, the angle XMP, the astronomical latitude of P.

ϕ', the angle XOP, its geocentric latitude.

x, y, the rectangular coordinates, OQ and QP, of P referred to the principal axes OX and OY.

ρ, the radius vector OP.

a, b, the major and minor semi-axes, OX and OY.

e, the eccentricity of the meridian XY.

The quantities supposed known are the dimensions and form of the geoid, expressed by a, b, and e, and the astronomical latitude of the place, found by direct observation. The quantities required for parallax are ρ and ϕ'. We adopt the usual notation and formulae of analytic geometry. From the equation of the normal it follows that the angle which ZP makes with the major axis is given by the equation

$$\frac{y}{x} = \tan \phi' = \frac{b^2}{a^2} \tan \phi. \qquad (1)$$

From the equation of the ellipse we have

$$b^2x^2 + a^2y^2 = a^2b^2, \qquad (2)$$

from which we are to determine x and y in terms of ϕ.

From the equation (1) we derive by multiplication by $a^2x\cos\phi$,

$$b^2x \sin \phi = a^2y \cos \phi$$

or

$$b^4x^2\sin^2\phi - a^4y^2\cos^2\phi = 0.$$

From this equation and (2) we find

$$x^2 = \frac{a^4\cos^2\phi}{a^2\cos^2\phi + b^2\sin^2\phi},$$

$$y^2 = \frac{b^4\sin^2\phi}{a^2\cos^2\phi + b^2\sin^2\phi}.$$

Introducing the eccentricity by the substitution $b^2 = a^2(1-e^2)$, we have

$$a^2\cos^2\phi + b^2\sin^2\phi = a^2(1-e^2\sin^2\phi).$$

Thus, introducing ρ and ϕ', we have

$$\left.\begin{aligned} x &= \rho\cos\phi' = \frac{a\cos\phi}{\sqrt{1-e^2\sin^2\phi}} \\ y &= \rho\sin\phi' = \frac{a(1-e^2)\sin\phi}{\sqrt{1-e^2\sin^2\phi}} \end{aligned}\right\} \quad \text{......(3)}$$

To find the angle of the vertical, $\phi - \phi'$, we use

$$\sin(\phi' - \phi) = \cos\phi\sin\phi' - \sin\phi\cos\phi'.$$

Substituting for $\sin\phi'$ and $\cos\phi'$ their values derived from (3) and noting that (3) gives

$$\frac{\rho^2}{a^2} = \frac{1-(2e^2-e^4)\sin^2\phi}{1-e^2\sin^2\phi}, \quad \text{......(4)}$$

we find

$$\sin(\phi - \phi') = \frac{1}{2}\frac{a}{\rho}\frac{e^2\sin 2\phi}{\sqrt{1-e^2\sin^2\phi}} = \frac{1}{2}\frac{e^2\sin 2\phi}{\sqrt{1-(2e^2-e^4)\sin^2\phi}}. \quad \text{...(5)}$$

76. Dimensions and compression of the geoid.

Instead of the eccentricity of the terrestrial meridian it is common to use its *compression* or *ellipticity*. By this is meant the fraction by which the ratio of the semi-axes differs from unity. Putting c for this quantity, we have

$$c = \frac{a-b}{a} = 1 - \sqrt{1-e^2}.$$

The numerical value of the compression is still somewhat uncertain owing to the small extent of the earth's surface over which precise geodetic measures have been extended. Indeed, from the very nature of the case, the compression must be a somewhat indefinite quantity, there being no one spheroid which we can exactly define as fitting the surface of the geoid better than any other.

The dimensions of the geoid as determined by Bessel many years ago are

$$a = 6\ 377\ 397 \text{ metres} = 6\ 974\ 532 \text{ yards},$$
$$b = 6\ 356\ 079 \text{ metres} = 6\ 951\ 218 \text{ yards}:$$

whence
$$c = \frac{1}{299{\cdot}15},$$
$$e = 0{\cdot}081\ 696\ 7.$$

These numbers have been generally used in astronomy and geodesy for the greater part of a century. During that interval geodetic measures have been greatly extended. A general determination made by Clarke of England from geodetic measures is

$$a = 6\ 378\ 249 \text{ metres},$$
$$b = 6\ 356\ 515 \text{ metres}:$$

whence
$$c = \frac{1}{293{\cdot}5},$$
$$e = 0{\cdot}08248.$$

Clarke's investigations also shew that the actual figure could be a little better represented by an ellipsoid with three unequal axes, the equator itself having a slight ellipticity. It is probable, however, that this apparent ellipticity of the equator arises from the irregularities with which the actual figure of the earth is affected.

As yet, geodetic measures cover so small a fraction of the earth's surface that an accurate determination of the compression cannot be derived from them. Measures of the force of gravity, as given by the length of the seconds' pendulum, are therefore still most relied upon for the purpose in question. It is, therefore, considered by the best authorities that Bessel's value of the compression is nearer the truth than Clarke's. Helmert, from a study of all the data, has recently derived numbers which will be found in Appendix I., and which may be regarded as the best yet reached.

Section II. Parallax and Semi-diameter.

77. The word *parallax*, in its most general sense, means the difference between the directions of an object as seen from two different points. If O (Fig. 15) be the object, and P and Q the points of observation, the parallax is the difference between the directions PO and QO. Its magnitude is measured by the angle $P'OQ' = POQ$ between the lines from P and Q to O.

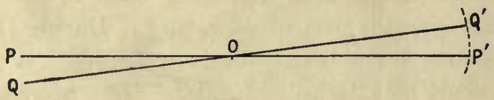

Fig. 15.

When used without any other qualifying adjective, *parallax* commonly means the difference in the directions of a heavenly body as seen from the point of reference, which may be the centre of the earth or of the sun, and from some point of observation on the surface of the earth.

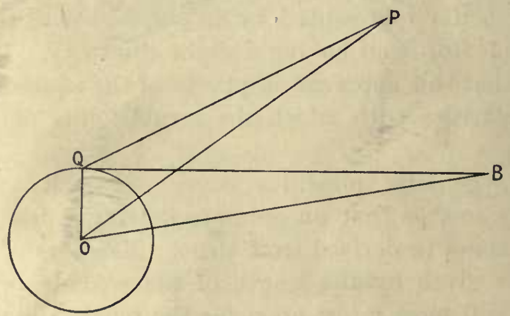

Fig. 16.

Parallax in altitude is the difference between the geocentric and apparent altitude of a body. If in Fig. 16, P is the body, Q the point of observation, and O the centre of the earth, the parallax in altitude is the angle OPQ.

If Q is so situated that the body is in its horizon, say at B, the parallax OBQ is called the *horizontal parallax*.

If also the point Q is on the earth's equator, so that OQ is the equatorial radius of the earth, the angle OBQ is called the *equatorial horizontal parallax* of the body.

It will be seen that the horizontal parallax is equal to the semi-diameter of the earth as seen from the body.

By *annual parallax* is meant the parallax when the point of reference is the sun and that of observation the earth.

By parallax in any coordinate is meant the difference between the values of that coordinate when referred to the centre of the earth as the origin, and when referred to a point on its surface. Thus we have parallax in R.A., in Dec., in Latitude, in Longitude, etc.

The horizontal parallax is connected with the radius of the earth, OQ, and the geocentric distance, OB of the body, by a simple relation. If we put

ρ, the radius of the earth at the point of observation;
r, the geocentric distance OB;
π_h, the horizontal parallax of B,
π_1, the equatorial horizontal parallax,

we have
$$r = \rho \sin \pi_h.$$

Hence
$$\sin \pi_h = \frac{\rho}{r}. \quad \text{.......(1)}$$

To express ρ and r in terms of the same unit of length, we remark that, in case of a planet, r is expressed in terms of the earth's mean distance from the sun, while ρ is commonly expressed in terms of the equatorial radius of the geoid. Hence, if ρ and r in (1) are expressed in this way, their quotient in (1) must be multiplied by the ratio of the two units, which is the sine of the sun's mean equatorial horizontal parallax $\equiv \pi_\odot$. We may then write instead of (1)

$$\pi_h = \frac{\rho}{r}\pi_\odot, \quad \text{.......(1')}$$

For the equatorial horizontal parallax of the planet, which is given in the ephemeris, we write 1 for ρ and π_1 for π_h in (1).

In the astronomical ephemerides the equatorial horizontal parallaxes of the principal bodies of the solar system are given. They are connected with the distance of the body by the relation

$$r \sin \pi_1 = a, \qquad \text{.............................(2)}$$

a being the equatorial radius of the geoid, expressed in the same unit as r.

78. Parallax in altitude.

There being two radii vectors of the body, one from the observer and one from the earth's centre, and two zeniths, there are, in all, four altitudes and zenith distances to be distinguished. We shall term a Z.D. measured from the geocentric zenith Z' (Fig. 14) *a reduced* Z.D., and one defined by the radius vector from the earth's centre a *geocentric* Z.D.

The effect of parallax is evidently to make the apparent greater than the geocentric Z.D., the azimuth when referred to that zenith Z' being unchanged. When a body is on the meridian the geocentric and apparent zenith lie on the same great circle with it, and the parallax has the same effect on the reduced and the apparent Z.D. But, if the body is not on the meridian, the displacement by parallax will not take place on a vertical circle, and both the altitude and azimuth of the body will be changed by it. The rigorous determination of the parallax in altitude and azimuth requires the solution of a spherical quadrangle of which the vertices are the two zeniths, and the geocentric and apparent positions of the body. The cases in which this solution is necessary are so rare that they need not be considered here. Parallax in altitude is commonly required only in the case of a body on the meridian.

To find the parallax in altitude on the meridian, we put

π_1, the equatorial horizontal parallax.

π_a, the parallax in altitude.

v, the angle of the vertical, taken positively in the northern hemisphere.

z, the apparent zenith distance of the body, positive toward the south.

z', the reduced Z.D.

Then

$$z' = z - v.$$

From the definitions already given and the constructions in Figs. 14 and 16, we have the following relations between the parallaxes and the geocentric distance r of the heavenly body:

From (2); $$\sin \pi_1 = \frac{a}{r}.$$

From (1); $$\sin \pi_h = \frac{\rho}{r} = \frac{\rho}{a} \sin \pi_1.$$

In the triangle OQP,

$$\text{angle } OQP = 180° - z';$$
$$\text{angle } OPQ = \pi_a;$$
$$\text{side } OQ = \rho;$$
$$\text{side } OP = r;$$

whence, by the law of sines,

$$\frac{\rho}{r} = \frac{\sin \pi_a}{\sin z'};$$
$$\sin \pi_a = \sin z' \sin \pi_h.$$

Whence we have for the parallax in altitude

$$\sin \pi_a = \frac{\rho}{a} \sin(z - v) \sin \pi_1. \qquad (3)$$

This, being subtracted from the apparent Z.D. gives the geocentric Z.D.

In the computation of parallaxes we take the equatorial radius of the earth as unity, and use the symbol ρ to designate the ratio of the radius vector of the place to the equatorial radius.

Thus the preceding expression becomes

$$\sin \pi_a = \rho \sin \pi_1 \sin(z - v). \qquad (4)$$

In the case of all heavenly bodies except the moon we may assume $\sin \pi$ to be identical with π itself.

79. Parallax in right ascension and declination.

To determine the difference between the right ascension or declination of a body as seen from the centre and from the surface of the earth, we first express the positions of the observer and of the body in rectangular coordinates, the origin being at the centre of the earth and the axes of reference as follows:

Z, the axis of rotation of the earth, positive toward the north.

X, the equatorial radius of the earth in the meridian of the observer. This axis cuts the celestial equator on the meridian.

Y, an equatorial radius cutting the earth's surface 90° west from the axis of X. This axis cuts the celestial sphere in the west point of the horizon of the place. Its positive direction is the opposite of the conventional one, in order to correspond to the usual measure of the hour-angle.

Then putting h for the west hour angle of the body and r and δ for its geocentric distance and declination, we have the following expression for its rectangular coordinates:

$$\left.\begin{aligned} x &= r\cos\delta\cos h \\ y &= r\cos\delta\sin h \\ z &= r\sin\delta \end{aligned}\right\} \qquad (5)$$

From the definitions of the coordinates just given the observer lies in the plane XZ. His coordinate z is that which, in treating the figure of the earth, we called y. Putting ξ, η, ζ for his coordinates referred to the present system of axes, we have:

$$\left.\begin{aligned} \xi &= \rho\cos\phi' \\ \eta &= 0 \\ \zeta &= \rho\sin\phi' \end{aligned}\right\} \qquad (6)$$

The last quantities are determined from the latitude of the observer, as already shewn. Putting x', y', z' for the coordinates of the body relative to the observer, we then have:

$$x' = x - \xi,$$
$$y' = y - \eta,$$
$$z' = z - \zeta.$$

We also distinguish the distance r', and hour-angle h' of the body as affected by parallax, by accents. Then substituting in the last equations for the rectangular coordinates their expressions in (5) and (6), we have

$$\left.\begin{aligned} r'\cos\delta'\cos h' &= r\cos\delta\cos h - \rho\cos\phi' \\ r'\cos\delta'\sin h' &= r\cos\delta\sin h \\ r'\sin\delta' &= r\sin\delta - \rho\sin\phi' \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(7)$$

The geocentric coordinates, r, δ, and h, being given, we could from these equations compute r', δ', and h', the corresponding coordinates relative to the observer. But it will be easier to compute the parallax in R.A. (or hour-angle) and Dec., or the values of $h - h'$ and $\delta' - \delta$. The problem may have either of the following two forms:

1. *Given*, the geocentric coordinates; to find the apparent ones.
2. *Given*, the apparent coordinates; to find the geocentric ones.

We treat the problem in the first of these forms. There are also two methods of solution: one when the parallaxes are so small that their second powers may be neglected; the other when this is not the case. The first of these is the case for all heavenly bodies except the moon. For the latter the solution should be rigorous.

80. Transformed expression for the parallax.

We transform the first two of equations (7) as follows. Multiplying the first by $\cos h$, the second by $\sin h$, and adding the products, we have the first of the following equations; multiplying the first by $\sin h$, and the second by $\cos h$, and taking the difference of the products we have the second.

$$\left.\begin{aligned} r'\cos\delta'\cos(h'-h) &= r\cos\delta - \rho\cos\phi'\cos h \\ r'\cos\delta'\sin(h'-h) &= \rho\cos\phi'\sin h \end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(8)$$

We shall now put

$\Delta\alpha = \alpha' - \alpha = h - h'$, the parallax in R.A.

$\Delta\delta = \delta' - \delta$, the parallax in Dec.

$f = \dfrac{r'}{r}$, the ratio of the distance of the body from the observer to that from the earth's centre. f is a little less than unity, being always contained between the limits 0·98 and 1.

Divide the equations (6) and (7) or (8) by r, and note that, taking the earth's equatorial radius as unity, we have

$$r=\frac{1}{\sin\pi_1}.$$

π_1, the equatorial horizontal parallax, is taken as given, and is found in the Ephemeris.

Putting for brevity

$$\left.\begin{aligned}\xi'&=\xi\sin\pi_1=\rho\cos\phi'\sin\pi_1\\ \zeta'&=\zeta\sin\pi_1=\rho\sin\phi'\sin\pi_1\end{aligned}\right\}, \qquad (9)$$

the equations (8) and $(7)_3$ become

$$\left.\begin{aligned}f\cos\delta'\sin\Delta\alpha&=-\xi'\sin h\\ f\cos\delta'\cos\Delta\alpha&=\cos\delta-\xi'\cos h\\ f\sin\delta'\qquad&=\sin\delta-\zeta'\end{aligned}\right\} \qquad (10)$$

The quotient of the first two equations gives

$$\tan\Delta\alpha=\frac{-\xi'\sin h}{\cos\delta-\xi'\cos h}.$$

If we compute

$$p=\xi'\sec\delta, \qquad (11)$$

this equation becomes

$$\tan\Delta\alpha=\frac{-p\sin h}{1-p\cos h}, \qquad (12)$$

which is easily computed by a table of addition and subtraction logarithms, especially that of Zech. It may be yet easier to use the principal table of Appendix IV., the form (12) being identical with that for the precession of a star in R.A. when we replace h by α, and assign a suitable value to p. For this purpose we compute

$$p_s=[4{\cdot}138\,334]\,\rho\cos\phi'\sin\pi_1\sec\delta,$$

enter the table with Arg. $p_s\cos h$, and take out K.

Then $$\Delta_t\alpha=Kp_s\sin h,$$
$$\Delta\alpha=\Delta_t\alpha-\text{red. from tangent to arc.}$$

In the case of the declination we may, instead of computing the parallax, compute δ' directly from the equations (10). The quotient $(10)_3\cos\Delta\alpha\div(10)_2$ is

$$\tan\delta'=\frac{(\sin\delta-\zeta')\cos\Delta\alpha}{\cos\delta-\xi'\cos h} \qquad (13)$$

This direct computation of δ' requires fewer separate quantities than that of the parallax; but 7-figure logarithms will be required to assure the result being correct to $0''\cdot 1$ whenever $\delta > 10°$. As 5-place logarithms only are required for the parallax, it will generally be easier to compute the latter.

To derive the formulae for the parallax in Dec. the simplest formulae for computation are derived by multiplying the first two equations (10) by $\sin \frac{1}{2}\Delta\alpha$ and $\cos \frac{1}{2}\Delta\alpha$ respectively, and adding. We thus derive the second of the following equations, the first being $(10)_3$.

$$\left.\begin{array}{l} f \sin \delta' = \sin \delta - \zeta' \\ f \cos \delta' = \cos \delta - \sigma \end{array}\right\}, \qquad \qquad (14)$$

where we write for brevity,

$$\sigma = \xi' \cos(h - \tfrac{1}{2}\Delta\alpha) \sec \tfrac{1}{2}\Delta\alpha. \qquad \qquad (15)$$

By forming

$$(14)_1 \times \cos \delta - (14)_2 \times \sin \delta \text{ and } (14)_1 \times \sin \delta + (14)_2 \times \cos \delta,$$

and adding, we have

$$\left.\begin{array}{l} f \sin \Delta\delta = \sigma \sin \delta - \zeta' \cos \delta \\ f \cos \Delta\delta = 1 - \zeta' \sin \delta - \sigma \cos \delta \end{array}\right\}. \qquad \qquad (16)$$

To facilitate the logarithmic computation of these equations compute g and G from

$$\left.\begin{array}{l} g \sin G = \zeta' \\ g \cos G = \sigma \end{array}\right\}; \qquad \qquad (17)$$

we shall then have

$$\tan \Delta\delta = \frac{-g \sin (G - \delta)}{1 - g \cos (G - \delta)}, \qquad \qquad (18)$$

a form similar to (12).

81. Mean parallax of the moon.

The moon is so much nearer to us than any other body of the solar system that its parallax rests upon a different basis from that of the planets. The mean value of its parallax is called its constant of parallax. The ratio of the actual parallax at any moment to this constant is determined from theory with all the precision necessary in any case whatever. But the actual value of the parallax as tabulated may require to be increased or diminished in an appreciable ratio on account of possible error in the constant.

The constant in question has been determined by observations from different points on the earth's surface, especially at the observatories of Greenwich and the Cape of Good Hope. It is also determined by the theory of gravitation, the problem being at what distance the moon should be placed in order that it may revolve around the earth in its observed time of revolution, allowance being made for the disturbing action of the sun. In this form the problem is the original one attacked by Sir Isaac Newton when he inquired whether the moon would be held in her orbit at the observed distance by the gravitation of the earth, the latter diminishing as the square of the distance. The theoretical method now affords the most accurate measure of determining the distance and, therefore, the parallax of the moon. The best result of theory yet obtainable is:

$$\text{Constant of equatorial parallax} = 3422''{\cdot}63.$$

As it is the sine of the parallax which enters into the formulae, while arc is most conveniently used in the expression, it is common to use the sine of the constant instead of the constant itself, this sine being reduced to seconds. We then have

$$\text{Sine of constant of parallax} = 3422''{\cdot}47.$$

The actual sine is found by dividing this expression by 206 264″·8, the number of seconds in the radius unit, or multiplying by sin 1″.

In Hansen's tables of the moon, which have been most widely used during the past forty years in the computations of the ephemeris, the adopted value of the constant is:

$$\text{Constant of } \sin \pi = 3422''{\cdot}07.$$

If, therefore, the best value of the parallax is required, the value from the ephemeris should be increased by multiplying it by the factor 1·000 118. Instead of multiplying by this factor the mean value of the correction, $+0''{\cdot}40$, may be added to all the values of the equatorial horizontal parallax in the Ephemerides without an error exceeding $\pm 0''{\cdot}03$.

82. Parallaxes of the sun and planets.

As the parallax of the nearest planet, Venus, rarely exceeds 30″, quantities of the second order as to the parallax of the sun and planets may always be dropped, thus greatly simplifying the computation. Putting

π_1'', the equatorial horizontal parallax expressed in seconds of arc,

the equations (9) will become

$$\xi' = \rho \cos \phi' \pi_1'' \sin 1'',$$
$$\zeta' = \rho \sin \phi' \pi_1'' \sin 1''.$$

Substituting in (12) $\Delta\alpha'' \sin 1''$ for $\tan \Delta\alpha$ and dropping quantities in p^2, we have, instead of (12),

$$\Delta\alpha'' = -\rho \cos \phi' \sin h \sec \delta \pi_1'' \qquad (18)$$

for the parallax in R.A. expressed in seconds ot arc.

With the same abbreviation, the computation of the parallax in Dec. takes the form

$$\left.\begin{aligned} g \sin G &= \rho \sin \phi' \pi'' \\ g \cos G &= \rho \cos \phi' \pi'' \cos h \\ \Delta\delta'' &= -g \sin (G - \delta) \end{aligned}\right\}, \qquad (19)$$

which will be the parallax in Dec. expressed in seconds of arc.

83. Semi-diameters of the moon and planets.

No observations have yet shewn any deviation of the apparent disc of the moon from the circular form, local irregularities of the surface excepted. The figure of our satellite is, therefore, treated as spherical. The linear radius, R_M is commonly expressed by its ratio to the equatorial radius of the earth, R_E, and is called k. This quantity cannot be measured directly, but is derived from the observed angular semi-diameter of the moon, combined with the parallax, taken as known. Since the moon's parallax is the earth's semi-diameter seen from the moon, it follows that if we put S_0, the moon's angular semi-diameter at the distance corresponding to the constant of parallax, we shall have, for the ratio of the radii of the earth and moon,

$$\frac{R_M}{R_E} = \frac{\sin S_0}{\sin \pi_1}.$$

From very comprehensive recent discussions of occultations of stars by the moon, made by Struve, Peters, and Battermann, it is inferred that the best value of the moon's semi-diameter at the distance corresponding to the constant of parallax is

$$S_0 = 932''\cdot 57.$$

The corresponding value of k is

$$k = \frac{\sin S_0}{\sin \pi_1}.$$

Using $\sin \pi_1 = 3422\cdot 47''$, this gives

$$k = 0\cdot 272\,483.$$

If Hansen's constant of parallax is used, the result is

$$k = 0\cdot 272\,516.$$

In the case of a planet, if we put r for its linear radius at any point on the edge of its apparent disc, as seen from the earth, its apparent angular radius to that point is found by the same geometric construction as the horizontal parallax (§ 77). Putting s for the angular semi-diameter, the value of s as seen by an observer at the distance r from the centre of the planet is given by the equation

$$\sin s = \frac{R}{r} \qquad (20)$$

The figures of all the planets may be treated as ellipsoids of revolution, of which the eccentricity vanishes if the figure is spherical. Let us put R_a and R_b, the semi-axes of the ellipsoid. The linear radius of the planet at latitude ϕ will then be given with all necessary precision by the equation

$$R = R_a(1 - c\sin^2\phi), \qquad (21)$$

c being the compression

$$c = \frac{R_a - R_b}{R_a}.$$

The apparent disc is a spherical ellipse of which the major semi-axis is found by putting R_a for R in (20). To find the minor axis, and its position-angle with respect to the hour-circle passing through the planet, let us put

A, D, the R.A. and Dec. of the pole H of the planet's axis of rotation.

α, δ, those of the planet itself, whose position on the celestial sphere we call K.

P, the celestial pole of the earth.

In the spherical triangle PHK will then be

$$\text{Side } HK = 90° + \phi. \qquad (22)$$

Angle PHK = position-angle of minor axis relative to the hour-circle through K.

We assume as given the pole H, in terms of its R.A. and Dec. If given in ecliptical coordinates, these are to be converted into equatorial ones. Then, the solution of the triangle PHK so as to find the angle H and the side HK will give us ϕ and the position-angle of the axes of the apparent disc.

Practically 3-figure logarithms will suffice in the solution of the triangle.

The astronomical data usually given for determining semi-diameters of the planets are the apparent angular semi-diameters S_0 at some standard distance r_0, for which the unit of distance or the mean distance of the sun is generally taken. Whatever the value of r_0, we have, for the apparent semi-diameter

$$\sin s = \frac{r_0}{r} \sin s_0. \qquad (23)$$

In the case of the planets, s is so small that we may always use the semi-diameters themselves expressed in seconds of arc, instead of $\sin s$.

CHAPTER VII.

ABERRATION.

84. The observed and commonly accepted law of displacement of a star by aberration is this:

Let S be the position of a star at the moment when a ray of light leaves it, and E that of the earth when the ray reaches it.

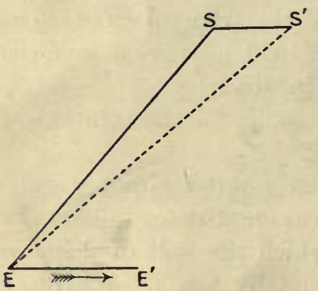

FIG. 17.

Let EE' be the direction in which the earth is moving at the moment, and v its velocity. Also, put V for the velocity of light.

Now draw SS' parallel to EE', and of such length that

$$SS' : SE = v : V.$$

Then the law in question is that the star S will be seen by an observer on E in the apparent direction ES'.

Stated in a general form the law is:

The apparent position of an object seen by an observer in motion is displaced from the true position in which it would be seen if the observer were at rest by an amount equal in linear measure to the observer's motion at constant speed during the time occupied by light in passing from the object to the observer. The direction of the displacement is that of the observer's motion at the moment of observation.

To express the law in algebraic form, put

v, the linear speed of the observer's motion.

ΔS, the apparent displacement in linear measure.

R, the distance of the object.

The time occupied by the light in passing will then be $R \div V$, and for the apparent linear displacement we have

$$\Delta S = \frac{v}{V} R.$$

The angular displacement of the object, represented by SES' is called its *aberration*; and its effect upon the value of any coordinate is called the aberration in that coordinate. To find its amount for any coordinate let us put

X, Y, Z, the rectangular coordinates of S referred to any system of axes having its origin in E;

x', y', z', the components of the velocity of the earth resolved in the direction of these axes;

ΔX, ΔY, ΔZ, the coordinates of S' relative to S, so that the coordinates of S' relative to E are

$$X + \Delta X; \quad Y + \Delta Y; \quad Z + \Delta Z.$$

To express ΔX, ΔY, ΔZ in terms of x', y', and z', let us put α, β, γ, the angles which the parallel lines EE' and SS' make with the coordinate axes. We then have

$$x' = v \cos \alpha, \quad \Delta X = SS' \cos \alpha = \frac{x' \Delta S}{v},$$

$$y' = v \cos \beta, \quad \Delta Y = SS' \cos \beta = \frac{y' \Delta S}{v},$$

$$z' = v \cos \gamma, \quad \Delta Z = SS' \cos \gamma = \frac{z' \Delta S}{v}.$$

Also R being the distance ES from the earth to the star,

$$\Delta S = \frac{v}{V} R.$$

We then have, by the preceding equations,

$$\left.\begin{aligned} \Delta X &= \frac{x'}{V} R \\ \Delta Y &= \frac{y'}{V} R \\ \Delta Z &= \frac{z'}{V} R \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

85. Reduction to spherical coordinates.

These are the general expressions for the apparent displacement of a star by aberration, and are valid for any rectangular system of axes. As the position of a star is always expressed by spherical coordinates, we must reduce the expressions to the corresponding ones in such coordinates.

Putting

L, the longitude of the star in any such system;

B, its latitude;

R, its distance,

we have

$$\left.\begin{aligned} X &= R\cos B\cos L \\ Y &= R\cos B\sin L \\ Z &= R\sin B \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The displacement by aberration is so slight, about 20″, that we may derive its value from these expressions by differentiation. We thus find, as in deriving formulae (4) of § 48,

$$\cos B\Delta L = -\sin L\frac{\Delta X}{R} + \cos L\frac{\Delta Y}{R},$$

$$\Delta B = -\sin B\cos L\frac{\Delta X}{R} - \sin B\sin L\frac{\Delta Y}{R} + \cos B\frac{\Delta Z}{R}.$$

Substituting for ΔX, ΔY, ΔZ, their values (1), we find for the effect of the aberration upon the spherical coordinates of the star,

$$\left.\begin{aligned} \cos B\Delta L &= -\frac{x'}{V}\sin L + \frac{y'}{V}\cos L \\ \Delta B &= -\sin B\cos L\frac{x'}{V} - \sin B\sin L\frac{y'}{V} + \cos B\frac{z'}{V} \end{aligned}\right\} \quad \ldots\ldots(3)$$

The next step is to substitute for x', y' and z', the resolved components of the motion of the earth, their expressions in terms of the elements of the earth's orbit. This requires the use of the elementary formulae for the elliptic motion of the earth round the sun, which we assume to be given.

We shall first take for L and B the ecliptical coordinates of the star. Let us put

λ, the longitude of the earth in its orbit;
r, the earth's radius vector;
π, the longitude of the earth's perihelion;
e, the eccentricity of its orbit;
f, its true anomaly;
n, its mean angular velocity around the sun;
λ', its actual angular velocity, or the value of $D_t\lambda$;
r', the value of $D_t r$;
x, y, z, the rectangular coordinates of the earth referred to the sun.

When referred to the ecliptic the motion of the earth in latitude is so minute that it may be left out of the problem. We therefore put

$$x = r\cos\lambda,$$
$$y = r\sin\lambda,$$
$$z = 0.$$

Then by differentiation

$$\left.\begin{aligned} x' &= r'\cos\lambda - r\lambda'\sin\lambda \\ y' &= r'\sin\lambda + r\lambda'\cos\lambda \\ z' &= 0 \end{aligned}\right\} \quad \dots\dots\dots\dots(4)$$

By the law of elliptic motion of a planet, it is shown that r, r', and λ' are given by the equations

$$\frac{a}{r} = \frac{1 + e\cos(\lambda - \pi)}{\cos^2\phi},$$

$$\lambda' = \frac{a^2}{r^2} n\cos\phi,$$

$$r' = \frac{aen\sin(\lambda - \pi)}{\cos\phi},$$

where ϕ is the angle of eccentricity, defined by the equation

$$\sin\phi = e,$$

whence $$\cos\phi = \sqrt{1 - e^2}.$$

Substituting these values of λ' and r' in (4), we find, by suitable reductions,

$$\left.\begin{aligned} x' &= -an\sec\phi(\sin\lambda + e\sin\pi) \\ y' &= \quad an\sec\phi(\cos\lambda + e\cos\pi) \end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots\ldots(5)$$

Substituting these values in the equation (3), and putting for brevity

$$\kappa = \frac{an\sec\phi}{V}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

we find

$$\left.\begin{aligned} \cos B\Delta L &= \kappa(\sin\lambda + e\sin\pi)\sin L \\ &\quad + \kappa(\cos\lambda + e\cos\pi)\cos L \\ &= \kappa\cos(\lambda - L) + e\kappa\cos(\pi - L) \\ \Delta B &= \kappa\sin B\sin(\lambda - L) + e\kappa\sin B\sin(\pi - L) \end{aligned}\right\}. \quad \ldots\ldots(7)$$

Studying the last terms of these equations, it will be seen that they are independent of the earth's longitude, and functions of the elements of the earth's orbit and the coordinates of the star. The variation of these quantities is so slow, and the factor so minute, that, unless the star be in the immediate neighbourhood of the pole, the terms in question may be regarded as constant for several centuries. They may, therefore, be left out of consideration for the present, being included in the values of the coordinates of the star as determined by observation.

In the usual formulae for aberration we put

$\odot$, the true longitude of the sun, $= \lambda - 180°$.

The aberration in the longitude and latitude of a fixed star may therefore be expressed in the form

$$\left.\begin{aligned} \cos B\Delta L &= -\kappa\cos(\odot - L) \\ \Delta B &= -\kappa\sin B\sin(\odot - L) \end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots\ldots(8)$$

86. The constant of aberration and related constants.

The coefficient κ, which is called the *constant of aberration*, demands our special attention. From the definitions of a and n it follows that an is the linear velocity which the earth would have if it moved in a circular orbit of radius a. As the earth actually moves, $an\sec\phi$ is the half-sum of its greatest and least velocities, which we may term (though not with strict correctness) its *mean velocity*. Thus from (6):

The constant of aberration is the ratio of the mean velocity of the earth in its orbit to the velocity of light.

This mean velocity is the product of its velocity were its orbit circular and its time of revolution unchanged, into sec ϕ.

There are two ways in which we may determine the constant of aberration:

(1) By observation of the annual change in the R.A. and Dec. of the stars produced by aberration. By the most refined measures yet made the constant is found to be $20''\cdot52$, with an uncertainty of 2 or 3 hundredths of a second.

(2) Supposing the dimensions of the earth's orbit to be known, we may compute the velocity of the earth. We have also determined, by actual measurement, the velocity of light. Thus the ratio of the two velocities may be computed. Let us put

$\pi_\odot$, the mean equatorial horizontal parallax of the sun;
ρ, the earth's equatorial radius.

To compute the mean velocity of the earth in its orbit, retaining $\pi_\odot$ as an unknown quantity, we have the data:

$$a = \frac{\rho}{\sin \pi_\odot} = \frac{6378\cdot2 \text{ kil.}}{\sin \pi_\odot}.$$

Taking one second as our unit of time, we have:

$$\text{Sidereal year} = 365 \text{ d. } 6 \text{ h. } 9 \text{ m. } 9 \text{ s.} = 31\,558\,149 \text{ s.},$$

$$n = \frac{\text{circumf.}}{\text{sid. year}} = \frac{6\cdot283\,19}{31\,558\,149},$$

$$\log \sec \phi = 0\cdot000\,061,$$

$$an \sec \phi = \frac{[7\cdot103\,83]}{\sin \pi_\odot} = \frac{[2\cdot418\,25]}{\pi''}.$$

Here, the number in brackets is the logarithm of the number to be used; and π'' means $\pi_\odot$ expressed in seconds of arc. The second fraction is derived from the first by multiplying its terms by the number of seconds in radius ($206\,265''$).

The measurement of the velocity of light gives the result

$$V = 299\,860 \text{ kilometres per second.}$$

Hence $$\kappa = \frac{[6{\cdot}941\,34]}{\pi''}, \quad\quad (9)$$

or, if we express κ in seconds, by multiplication by 206 265,

$$\kappa = \frac{[2{\cdot}255\,76]}{\pi''} = \frac{180{\cdot}20}{\pi''}.$$

We thus have, between κ'' and π'' the fundamental relation

$$\kappa'' \times \pi'' = 180{\cdot}20. \quad\quad (10)$$

We have retained π'' as an unknown quantity, because it is very difficult to determine, whereas the number 180·20 is probably correct within 3 or 4 units of its second place of decimals.

It follows that, of the constant of aberration and the solar parallax, we can determine the one when we know the other. They can be determined by observation with perhaps equal absolute accuracy, but as κ'' is more than twice as great as π'', this implies that it is determined with greater relative accuracy. The solar parallax can, therefore, be determined from κ with more accuracy than in any other one way, if we admit the completeness of the fundamental theory of aberration.

87. Aberration in right ascension and declination.

This may be determined by referring the position of the star and the motion of the earth to equatorial coordinates, which are those most used in computations relating to the fixed stars. Let us put

x_1, y_1, z_1, the heliocentric coordinates of the earth referred to the equatorial system.

x'', y'', z'', the corresponding velocities.

The transformation from the ecliptic system to the equatorial system is found by writing in the equations (7) or (8) of § 49,

$$x'', y'', z'', \text{ for } x, y, z,$$
$$x', y', z', \text{ for } X, Y, Z,$$

which gives
$$x'' = x',$$
$$y'' = y' \cos\epsilon - z' \sin\epsilon,$$
$$z'' = y' \sin\epsilon + z' \cos\epsilon.$$

We thus find from (5), substituting $\odot+180^\circ$ for λ, and taking π to represent the longitude of the solar perigee,

$$\pi = 281^\circ\ 13' \text{ in } 1900;$$

$$\left.\begin{aligned} x'' &= \quad an\sec\phi\,(\sin\odot + e\sin\pi) \\ y'' &= -an\sec\phi\cos\epsilon(\cos\odot + e\cos\pi) \\ z'' &= -an\sec\phi\sin\epsilon(\cos\odot + e\cos\pi) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(11)$$

If we also put X_1, Y_1, Z_1, the rectangular coordinates of the star referred to the equatorial system, the equations (1) give for its displacement

$$\frac{\Delta X_1}{R} = \frac{x''}{V} = \kappa\,(\sin\odot + e\sin\pi),$$

$$\frac{\Delta Y_1}{R} = \frac{y''}{V} = -\kappa\cos\epsilon\,(\cos\odot + e\cos\pi),$$

$$\frac{\Delta Z_1}{R} = \frac{z''}{V} = -\kappa\sin\epsilon\,(\cos\odot + e\cos\pi).$$

For the reason already mentioned we may leave out of account the constant terms $e\kappa\cos\pi$ and $e\kappa\sin\pi$, and write

$$\left.\begin{aligned} \frac{\Delta X_1}{R} &= \kappa\sin\odot \\ \frac{\Delta Y_1}{R} &= -\kappa\cos\epsilon\cos\odot \\ \frac{\Delta Z_1}{R} &= -\kappa\sin\epsilon\cos\odot \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

The effect on the R.A. and Dec. of the star, when quantities of an order higher than the first are dropped, is found by putting in (4a) of § 48, Δ, X_1, Y_1, Z_1, and R for d, x, y, z, and r.

$$\left.\begin{aligned} \cos\delta\Delta\alpha &= -\kappa\cos\epsilon\cos\odot\cos\alpha - \kappa\sin\odot\sin\alpha \\ \Delta\delta &= \kappa\cos\epsilon\cos\odot\sin\delta\sin\alpha - \kappa\sin\odot\sin\delta\cos\alpha \\ &\qquad - \kappa\sin\epsilon\cos\odot\cos\delta \end{aligned}\right\} \quad \ldots\ldots\ldots(13)$$

The form in which these equations are used in practice will be shown in the chapter on the reduction of places of the fixed stars.

88. Diurnal aberration.

In the preceding theory the motion taken into account has been only that of the centre of the earth around the sun. But, in consequence of the earth's rotation on its axis, the observer is continually carried toward the east point of his horizon with a speed, in metres per second,

$$v = 464\,\rho\cos\phi', \qquad\qquad (14)$$

ϕ' being his geocentric latitude, and ρ the radius of the earth at his station in terms of the equatorial radius. The speed of rotation of a point on the earth's surface at the equator is 464.

The effect of this motion is to produce a universal displacement of the apparent positions of all the bodies in the heavens toward the east point of the horizon, on great circles passing through this point, expressed by

$$s = \frac{v}{V}\sin\theta,$$

θ being the distance of the body from the east point. Putting for v and V their numerical values and reducing to seconds,

$$s = 0''\cdot 319\,\rho\cos\phi'\sin\theta. \qquad\qquad (15)$$

To find the effect of the displacement upon the R.A. and Dec. of the body, consider the spherical triangle PES formed by the pole P, the east point of the horizon, E, and the body S. Then θ is the side SE; and if we put

q, the angle at S;

h, the hour-angle of the body,

the aberration in R.A. and Dec. will be:

$$\left.\begin{aligned}\cos\delta\Delta\alpha &= s\sin q\\ \Delta\delta &= s\cos q\end{aligned}\right\} \qquad\qquad (16)$$

We have, in the triangle,

$$\sin\theta\sin q = \cos h$$
$$\sin\theta\cos q = \sin\delta\sin h.$$

Substituting in (16) the value of s from (15), we find

$$\left.\begin{aligned}\Delta\alpha &= 0''\!\cdot\!319\,\rho\cos\phi'\cos h\sec\delta\\ \Delta\delta &= 0''\!\cdot\!319\,\rho\cos\phi'\sin\delta\sin h\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots(17)$$

In using this expression, we may put $\rho=1$ and $\phi'=\phi$.

In reducing meridian observations due correction is made for the effect of diurnal aberration. Generally, however, it is ignored in practical astronomy, because it affects all bodies in the same region by the same amount: and that amount being very minute is seldom of practical importance. It should, however, be taken into account in all investigations involving the relative positions of widely separated bodies.

89. Aberration when the body observed is itself in motion.

The preceding theory is based on the relations of an observer in motion to a ray of light which has emanated from a heavenly body, the possible motions of that body being left out of consideration. Since the course of the ray depends wholly upon the position of the emitting body at the moment when the ray left it, and is independent of the position of that body at any other moment, the theory already developed is complete for the position of the body at the moment in question. That is to say, when the correction for aberration is applied to the apparent position of a body, the result will not be the position of the body at the moment T of observation, but at the moment $T-\tau$, τ being the time required for light to come from the body to the observer. If, therefore, the actual direction of the body is required for the time T of observation, its motion during the interval τ must be determined and added.

The general theory sets no limitations upon the motion of the body during the interval occupied by the passage of the light to our system. A double star revolving in an orbit may make several revolutions during this interval. In stellar astronomy generally no account is taken of these possible motions or changes. The fact that the distance of the stars, and therefore the time τ, is not known with precision, prevents any accurate determination of the motion during this interval, and at the same time renders it unimportant. All our statements respecting what is going on among the stars at a stated time T really refer to phenomena which occurred at a time $T-\tau$, τ being an unknown number of years, which we regard as constant for any one star or system, and which is left out of consideration.

When the forces which may possibly be at play among the stars admit of more complete investigation than they now do, it may be that the variations during the interval τ will enter as important elements into the problem. It is interesting, if not essential, to remark that, from the best estimate that can be made of the distance of the star 1830 Groonbridge, its actual direction is about 3′ ahead of its observed and adopted position, in the direction of its proper motion.

90. Case of rectilinear and uniform motion.

When the motion of the observed body is rectilinear and uniform during the time τ, it can be shown that its displacement by aberration depends solely upon the relative motions of the observer and the body, irrespective of the absolute motions of either. To show this, let us put

X', Y', Z', the components of the speed of the body in the direction of the three coordinate axes.

X_0, Y_0, Z_0, its coordinates at the moment $T-\tau$ when the light left it by which it was observed at the time T. Its actual coordinates at this time are found by adding to X_0, Y_0, Z_0, the motion during the time τ, and, therefore, are

$$X = X_0 + X'\tau,$$
$$Y = Y_0 + Y'\tau,$$
$$Z = Z_0 + Z'\tau.$$

Its apparent coordinates are expressed by adding to X_0, Y_0, and Z_0 the displacements given by the equations (1), in which we have

$$R = V\tau. \quad\quad (18)$$

These coordinates are therefore

$$X_{ap} = X_0 + x'\tau,$$
$$Y_{ap} = Y_0 + y'\tau,$$
$$Z_{ap} = Z_0 + z'\tau.$$

The differences between the true and apparent coordinates at the epoch T now become

$$\left.\begin{aligned} X_{ap}-X&=(x'-X')_T \\ Y_{ap}-Y&=(y'-Y')_T \\ Z_{ap}-Z&=(z'-Z')_T \end{aligned}\right\}, \quad \text{.....................(19)}$$

which depend only on the differences

$$x'-X', \text{ etc.}$$

The last numbers of these equations express the change in the coordinates of the body observed, relative to the moving earth as an origin, during the period occupied by light in passing from the body to the earth. The total displacement by aberration is therefore, in the case supposed, equal to this change. If, instead of rectangular coordinates, we use R.A. and Dec., the expression for the aberration in these coordinates will be

$$\left.\begin{aligned} \text{Ab. in R.A.} &= -\tau D_t\alpha \\ \text{Ab. in Dec.} &= -\tau D_t\delta \end{aligned}\right\}. \quad \text{.....................(20)}$$

To find τ, the distance R of the body must be known. If we express R and V in terms of the semi-major axis of the earth's orbit as unity, we shall have from (18)

$$\tau=\frac{R}{V},$$

which may be substituted for τ in the above expression.

91. Aberration of the planets.

In the astronomical ephemerides there are two systems of dealing with aberration of the planets. One consists in giving the apparent coordinates of the planets at the epochs of the ephemeris, commonly mean noon, as affected by the aberration. These apparent coordinates are found by applying the corrections (20) to the true coordinates.

As, for theoretical purposes, it may sometimes be desirable to have the actual position of the planet at the assigned time, and as, in the case of newly discovered objects, the distance may be unknown, the method is sometimes adopted of giving at the stated epochs the actual position of the planet. Then, if the distance is known, the value of τ can be computed. If from such an ephemeris apparent coordinates of a planet are required at a given time T, we subtract τ from T and interpolate the true coordinates to the moment $T-\tau$. This gives us the apparent position of the planet affected by aberration at the time T.

CHAPTER VIII.

ASTRONOMICAL REFRACTION.

Section I. The Atmosphere as a Refracting Medium.

92. Astronomical refraction is the refraction of a ray of light by the atmosphere as it is passing from a celestial object to the eye of the observer. Its measure is the change produced by it in the direction of the ray. The total amount of refraction depends, not only upon the density of the various strata of air through which the ray passes, but also on the direction of the strata of equal density with respect to the vertical. When the atmosphere is in a condition of equilibrium, these strata are horizontal. But, owing to aerial currents and other causes, this is not universally the case. The deviation from horizontality is specially marked in the strata which separate the air inside an observing room, and even inside the tube of a telescope, from the external air. The refraction due to this cause belongs to the subject of practical and instrumental astronomy, and therefore will not be considered in the present chapter. For the most part the astronomer is under the necessity of neglecting all irregularities in the density of the air, and considering the strata as horizontal, for the reason that it is seldom practicable to determine the effect of such irregularities with precision.

The general theory of astronomical refraction, as it will be set forth in the present chapter, therefore rests on the assumption that in the air the strata of equal density, or the equiponderant strata, are horizontal; and that the density continually diminishes from the earth to the outer limit of the atmosphere, subject to the conditions of equilibrium. What that limit may be is not yet known; we can, however, say with confidence that above a height of 60 kilometres the atmosphere is so rare that its refractive power need not be investigated.

Refraction may be treated in three sections. In the present section the law of density of the air on which its refractive power depends will be developed. In the next section a general conspectus of the laws and results of refraction by the atmosphere will be put into an elementary form. The third section will treat the general theory of astronomical refraction proper.

93. Density of the atmosphere as a function of the height.

We put in this section

ρ, the density of the atmosphere at any height.

ρ_1, the density at the earth's surface, or at the point where the observer is situated.

ρ_0, the "standard density" under standard barometric pressure (760 mm. at Paris) and temperature 0° C.

g, the ratio of the intensity of gravity at any point to standard gravity.

p, the pressure of the air per unit of surface at any altitude.

h, altitude above the surface in linear measure.

τ, the temperature in degrees centigrade above absolute zero.

τ_0, the absolute temperature of the centigrade 0°.

It should be remarked that the temperature from which τ is counted is not strictly the absolute zero, but the temperature at which the volume of air would become zero, supposing it, when placed under constant pressure, to continue its diminution of volume with falling temperature at the same rate that it is observed to vary through the range of temperature at which refraction occurs. This variation of volume with temperature is assumed to be linear, in accordance with the law of Gay Lussac.

If we measure the coefficient of expansion by the expansion at 0° C. produced by a change of 1° C. in the temperature, the absolute zero as here used will be the negative reciprocal of the coefficient. The following four values of the coefficient and of the absolute zero are the results of numbers determined or adopted by different authorities. The first two, those of Regnault and Mendelejeff, are determined by experiments in the laboratory. The third is that adopted by Bessel in his tables of refraction. The fourth is that adopted in the Poulkova tables of refraction.

	Coef. of exp.	Abs. zero.
Regnault - - -	·003 670	−272°·5
Mendelejeff - - -	·003 684	−271 ·4
Bessel (adopted) - -	·003 643 3	−274 ·5
Poulkova (adopted) -	·003 689	−271 ·1.

The mean of the experimental determinations would be −271°·9. But, as the actual amount of the refraction may be affected by the aqueous vapour in the atmosphere, we shall give equal weight to the number adopted at Poulkova and to that derived by experiment, thus taking for the temperature in question

$$\text{Temp. C. of abs. zero} = -271^\circ{\cdot}5.$$

The expression for τ in terms of temperature C. is therefore

$$\tau = \text{temp. C.} + 271^\circ{\cdot}5 = \text{temp. C.} + \tau_0.$$

94. The following two laws of physics are accepted as the basis of the subject.

First law: The pressure per unit of surface due to the elasticity of the air is proportional to the product of the density of the air into its absolute temperature.

Second law: Along any vertical line in the atmosphere the diminution of the pressure through any height is equal to the weight of a stratum of air of unit base and of that height.

The first of these laws cannot be true for temperatures approaching the absolute zero, and the density also varies slightly from the law for very high temperatures. But these deviations are unimportant in the theory of refraction. At an altitude where there is any possibility of the temperature approaching absolute zero the air is so rare as to be without appreciable effect upon the refraction. On the other hand, astronomical observations are not made at any but ordinary temperatures, so that the deviation in the case of high temperatures need not be considered.

The second law when rigorously applied requires us to take account of the diminution of gravity with height, which makes the actual law of density somewhat different from what it would be were gravity equal at all heights. But, as we shall hereafter see, this deviation is without important influence upon the refraction.

The first law may be expressed in the form of the equation

$$p=\gamma\tau\rho, \qquad (1)$$

γ being a constant depending on the elasticity of air at a given temperature and density. The density ρ being the fundamental quantity on which the refraction depends, this law may be expressed for practical use in the form

$$\rho=\frac{p}{\gamma\tau}. \qquad (2)$$

In developing this subject it is important to define the units in which the various quantities are to be expressed. Since the element of time does not enter into the theory, and the element of mass enters only in a subsidiary rôle, the most convenient units will not be those of the C.G.S. system, but the following:

The unit of length: arbitrary.

The unit of volume: the cube whose side is the unit of length.

The unit of weight: the weight of unit volume of standard water under standard gravity. For standard gravity it will be convenient to take gravity at Paris.

The unit of pressure: the pressure of unit weight upon unit surface.

The result of this choice of units is that the constant γ and the pressure h may both be defined as lengths. It will be seen that the constant γ may be defined as the elastic pressure of the air at the absolute temperature 1° when compressed to the density of water. Assuming that the first law can be continued to this temperature, the constant may be yet better apprehended by considering it as a height of the column of standard water which, under gravity at Paris, would condense air at absolute temperature 1° to density 1.

It is also convenient to adopt a standard of temperature, to which all the fundamental data will be referred. The best standard for tables of refraction is one not differing greatly from the mean temperature at which astronomical observations are made. A convenient standard is 10° C. or 50° F. But, in investigating the theory of the subject, it is better to take 0° C., because this is the standard temperature for nearly all physical investigations and numerical constants which enter into the theory. When, therefore, we speak of 760 mm. of mercury as a standard pressure, we mean mercury at 0° C., to which we suppose the observed height to be reduced by correcting it for the temperature of the mercury.

As our units have just been defined, the weight w of any volume v of air will be given by the equation

$$w = gv\rho.$$

Now, consider a prism of air of unit surface and of infinitesimal height dh. The weight of this prism is $g\rho dh$. Hence, regarding the prism as horizontal, the change of pressure through an infinitesimal change in height is given by the equation

$$dp = -g\rho dh, \quad\text{.............................}(3)$$

the sign being negative because the pressure decreases as we ascend.

If we regard ρ as constant through the height h of a column of air, the pressure at the base of this column produced by its weight will be

$$p = g\rho h. \quad\text{.................................}(4)$$

A comparison of this equation with (1) will give us the height of a column of air which, under standard gravity, would produce the same pressure that is actually exerted by the entire body of the air at the base of the column. This height is called the *pressure-height*. Let us put

h_1, the pressure-height.

Substituting this value of h in (4) and comparing with (1), we see that the pressure height is given by the equation

$$h_1 = \frac{\gamma\tau}{g}. \quad\text{.................................}(5)$$

The three quantities ρ, p, and τ all vary with the height h above the surface, and are to be expressed as functions of h. Regarding them as such, the differentiation of (2) gives

$$\frac{d\rho}{dh}=\rho\left(\frac{dp}{pdh}-\frac{d\tau}{\tau dh}\right). \quad \text{.........................(6)}$$

By substituting the value of dp from (3), we now have, after simple reductions,

$$\frac{d\rho}{dh}=-\frac{\rho}{\tau}\left(\frac{g}{\gamma}+\frac{d\tau}{dh}\right), \quad \text{.........................(7)}$$

which is the fundamental equation for the diminution of the density of the air with its height.

95. Numerical data and results.

It will conduce to clearness if, before going farther, we consider the numerical values of the fundamental quantities.

According to Regnault, whose results we accept for this purpose, the density of air (water = 1) at 0° C. ($\tau=271°{\cdot}5=\tau_0$) and under a barometric pressure of 760 mm. (gravity at Paris) is

$$\rho_0=0{\cdot}001\,293\,2.$$

This value is substantially confirmed by the more recent ones of Rayleigh and of Leduc.

Taking the metre as the unit of length, the unit of pressure, as we have defined it, will be the weight at Paris of a cubic metre of water pressing upon a square metre of surface. The standard barometric pressure just cited will be

$$\text{Density of quicksilver}\times 0{\cdot}760.$$

Taking for this density 13·596, this product is 10·333, the standard pressure. Substituting this value of p, and the values of τ_0 and ρ_0 corresponding to the given conditions in the equation (1), we have

$$10{\cdot}333=\gamma\times 271{\cdot}5\times 0{\cdot}001\,293\,2,$$

which gives $$\gamma=29{\cdot}429 \text{ metres}. \quad \text{.........................(8)}$$

This value of γ being substituted in (2) and (7) will give ρ and $\frac{d\rho}{dh}$ in terms of p, g, and τ. It will, however, be convenient to substitute for p the height of the barometer, by which p is determined in practice. Let us then put

b, the barometric reading ÷ 760 mm. and corrected for temperature.

We shall then have $p = 10{\cdot}333bg$,

and the equations (2) and (7) may be written

$$\rho = {\cdot}001\,293\,2\,\frac{\tau_0}{\tau}\,bg, \quad\ldots\ldots\ldots\ldots(9)$$

$$= \frac{0{\cdot}351\,10}{\tau}\,bg,$$

$$\frac{d\rho}{dh} = -\frac{\rho}{\tau}\left(0{\cdot}033\,980\,g + \frac{d\tau}{dh}\right). \quad\ldots\ldots\ldots\ldots(10)$$

96. An interesting conclusion may be drawn from this last equation. If the rate of diminution of temperature with height is such that

$$\frac{d\tau}{dh} = -0{\cdot}033\,98g,$$

which, g never differing much from 1, implies a diminution at the rate of 1° in about 30 metres, ρ will remain constant, the effect of continually diminishing pressure being compensated by the diminishing temperature. If this constant rate of diminution continues we shall, at the pressure-height, have $\tau = 0$, at which point the atmosphere will cease. That is to say, the result will be that the atmosphere would become an ocean of uniform density with a definite upper surface.

97. General view of requirements.

The logical order of the requirements for our theory is this:

In order to determine the refraction at considerable zenith distances it is necessary to have an expression for the density of the air as a function of the height. This density does not admit of direct observation; it must therefore be inferred from the law of diminution of temperature with height. When this is known, the law of density is derived by substituting in the equation (7) the values of τ and of $\frac{d\tau}{dh}$, which will be functions of h. The integration of the equation will then give the expression for the density at any height.

The diminution of temperature with height is of such a character that it cannot be exactly represented by any formula. It varies with the time of day, the seasons, the temperature at the surface of the earth, and the height itself. All we can do, therefore, is to construct some hypothetical formula which will give a result as near as possible to the general average. This formula may be based partly on theoretical considerations showing what laws of diminution are more or less probable and on observations with the aid of kites and balloons. These observations have been greatly extended during the past few years, and the results enable us to formulate laws of temperature with much greater confidence than was formerly possible.

Some theoretical considerations will help to guide us. Were the atmosphere in a state of complete rest the general theory of heat leads to the conclusion that it would tend to assume the same temperature throughout. This tendency may be counteracted in one direction or another by the effect of possible thermal coloration of the air, a subject about which not enough is known to form the basis of a conclusion. A state of constant temperature has also been shown by Tait to result from the kinetic theory of gases. This state is therefore called one of *thermal equilibrium*.

But the atmosphere is not at rest, being subject to ascending and descending currents. Whenever a body of air ascends, it expands and thereby cools. When a body descends it is compressed into smaller space and thereby becomes warmer. If the air were constantly stirred from top to bottom, a condition would be reached in which, when any body of it ascended, it would, as it expanded and cooled, constantly be at the same temperature as the surrounding air. This condition is that of *adiabatic equilibrium*.

When an atmosphere in thermal equilibrium is stirred so as to bring it nearer the state of adiabatic equilibrium, work must be done. For whenever, in such a case, a body of air is lifted up, it will by its expansion be colder, and therefore denser, than the surrounding air. It will, therefore, tend to fall again, and work must be done in order to raise it. A body of air falling becomes warmer, and so rarer, than the surrounding air, thus tending to rise again. We conclude that the centre of gravity of a column of air is higher in the case of adiabatic than in that of thermal equilibrium.

The law of diminution of temperature in the case of adiabatic equilibrium is such that the diminution of temperature with height is at the rate of about 10° C. per kilometre.

If the only source from which the air obtained heat, or to which it communicated heat, were the ground on which it rests, the tendency of ascending and descending currents would be toward bringing about a condition of adiabatic equilibrium of temperature throughout. But this tendency is continually counteracted by the effect of the sun's rays, and by radiation from the warmer portions of the air to the cooler portions. It is a result of the laws of radiation in space that, at each distance from the sun, there is a certain definite temperature which a neutral coloured body, or a body for which the reflecting and absorbing powers were the same for all wave-lengths, would reach. This normal temperature could be fairly well determined, and has especially been investigated by Poynting, his conclusions being based on the law that the emission of heat by a warm body is proportional to the fourth power of its absolute temperature.

It follows that, except so far as influenced by thermal coloration, portions of the atmosphere which are below this normal temperature will be warmed toward it by absorption of the sun's rays, while those which are at a temperature above the normal will lose heat by their own radiation. Thus arises a tendency toward thermal equilibrium the exact extent of which cannot be determined by theory, but only by observation.

98. Density at great heights.

The probable upper limit of the atmosphere may be considered in this connection. Observations of meteors and shooting stars seem to show that these bodies are seen at an altitude of more than 100, possibly 200, miles. The conclusions from this would be that the rate of diminution of temperature must diminish, so that the absolute zero is never reached.

Another indication of the height of the atmosphere is afforded by twilight. This terminates when the sun is at a depression of between 15° and 18° below the horizon. This leads to the conclusion that the power of reflecting light ceases, owing to the rarity of the atmosphere, at a height of about 70 kilometres. This fact suggests that the variations of temperature of these high regions of the air at different latitudes and seasons is probably less than near the surface of the earth. Altogether theory can tell us little about the actual diminution of the temperature with height except that it is materially less than 10° C. per kilometre. We must, therefore, derive an empirical law of diminution from observations.

The mass of material here at our disposal is very great, and, belonging to the province of meteorology, cannot be considered in the present work. The most valuable of this material consists of observations made by kites or balloons carrying self-registering thermometers to the greatest possible height. For several years past extensive kite observations have been made at the Blue Hill Observatory, Hyde Park, Mass., by Dr. A. L. Rotch. An extended system both of kite and balloon observations is being carried out by the U.S. Weather Bureau. In Germany and France balloons have recently been successfully sent to a height of 12 or more kilometres. Without going into unnecessary details, it may be said that these observations, notwithstanding the irregularities naturally inherent in the subject, lead to the following general conclusions:

1. The annual and diurnal changes of temperature diminish with the height, both becoming very small at the highest attainable altitudes. This result points to the conclusion that the sun's rays have but little immediate influence on the temperature of the higher strata of the atmosphere. Another obvious result is that the fall of temperature must be more rapid the higher the temperature is at the surface of the earth.

2. The diminution of the diurnal variation of temperature is

very rapid near the earth's surface, being reduced to one half at an altitude of a few hundred metres. One result of this is that during the day the diminution is very rapid at low altitudes, but, during the night, especially the later hours of the night, is changed to an actual increase.

3. The general average diminution near the ground, day and night, taking the whole year round, is not far from $6^\circ\cdot5$ C. per kilometre. Astronomical observations being mostly made at night, the diminution for them will be less. Were the rate of diminution near the surface of the earth important, it would be necessary to suppose a very small rate in the lowest kilometre of the air for the purpose of computing the astronomical refraction for night observations. But, for reasons which will be better understood when the general theory is developed, astronomical refraction is little influenced by the diminution of temperature at low altitudes, the effect of differences of temperature reaching their maximum near the pressure-height, and slowly diminishing for yet greater heights. We must, therefore, for astronomical purposes, lay more stress on the temperature at considerable heights than near the surface of the earth. This will enable us to include an expression for all heights in some simple formula.

99. Hypothetical laws of atmospheric density.

The various tables of refraction which have been constructed for astronomical use rest upon different expressions for the density of the air as a function of the altitude. It will also be instructive to consider hypothetical laws of diminution which are the simplest in form. Amongst the various hypotheses that have been made, or may be made, the following are worthy of citation:

A, the hypothesis of constant temperature at all altitudes. This hypothesis was adopted by Newton, and is, therefore, associated with his name. The law to which it leads is extremely simple.

B, the hypothesis that the density diminishes uniformly with the height. This is one of the forms which a law proposed by

Bouguer, and adopted by Simpson and Bradley, may take. On this hypothesis the density becomes zero, and the atmosphere reaches its limit at twice the pressure-height, or an altitude of about 16 kilometres. This is obviously too low; yet the hypothesis gives a better approximation to the truth so far as refraction is concerned, than that of constant temperature.

C, *Bessel's Hypothesis.* This is a modified form of the hypothesis of Newton. It is not based on any assumed law of temperature, but is an expression for the density in terms of the altitude. As will be shown hereafter, it is not altogether admissible.

D, the hypothesis that the temperature diminishes at a uniform rate with the height at all heights. This is in accordance with the law of adiabatic equilibrium, and accords most nearly with the results of the highest balloon observations, which show no falling off in the rate of diminution at the great heights yet attained. But it is not accordant with the kite observations, which, in the general average, seem to show a falling off of the rate at an altitude of a very few kilometres. It was developed by Ivory, with whose name it may be associated.

On this hypothesis, using the most likely rate of diminution, the absolute zero would be reached, and the atmosphere would have a limit at a height of 50 kilometres more or less.

E, the hypothesis that the temperature diminishes by a constant fraction of its absolute amount for every unit of increase in the altitude. The three conditions which this law satisfies are that of a rate of diminution which shall be more rapid in warm weather than in cold, and which shall slowly diminish with increasing height. It fulfils the additional condition that the absolute zero shall not be reached at any finite altitude. Yet, it cannot be applied so as to be strictly consonant with the temperature resulting from balloon ascensions.

It may be remarked that the preceding hypotheses are not entirely distinct, A and B being really special cases of Ivory's hypothesis. When, in D, the diminution of temperature is reduced to zero, we have on this hypothesis a constant temperature, and therefore Newton's hypothesis. If the temperature diminishes at such a uniform rate as to reach the absolute zero at the pressure height, we shall have the law B. The truth probably lies between these two very wide extremes.

100. Development of the hypotheses.

So far as refraction is concerned, we can assign a rate of diminution on either of the last two hypotheses which shall bring their results into close agreement at moderate heights. When this is done it is probable that either of them will represent the law of refraction equally well. Our choice between them or our combination of them must, therefore, be a matter of convenience. We shall now show the formulae for density to which these various hypotheses lead:

A. Newton's hypothesis of constant temperature. Disregarding the diminution of gravity with altitude, and supposing the temperature constant (7) gives

$$\frac{d\rho}{\rho dh} = \frac{-g}{\gamma\tau} = -\frac{1}{h_1}.$$

By integration, $$\log \rho = C - \frac{h}{h_1},$$

C being the arbitrary constant of integration. Putting

ρ_1, the density at the surface of the earth, or at the point of observation,

we shall have $$C = \log \rho_1,$$

and $$\rho = \rho_1 e^{-\frac{h}{h_1}}. \qquad \text{(11)}$$

The best idea of this result will be gained by the consideration that it shows the density to diminish in geometrical progression as the altitude increases in arithmetical progression. A clear general conception of the law may be gained by finding an altitude h_0, at which the density is reduced to one half. We may take 8 kilometres as the usual value of h_1, the pressure-height. We then find h_0 by putting, in (11), $\rho = \frac{1}{2}\rho_1$; $h_1 = 8$.

$$e^{\frac{h_0}{8}} = 2,$$

which gives $$h_0 = 8 \text{ Nap. } \log 2 = 5{\cdot}54,$$

h_0 being expressed in kilometres. The density would, therefore, be reduced one-half for about every $5\frac{1}{2}$ kilometres, or $3\frac{1}{2}$ miles.

Since $2^{10}=1024$ the density is reduced to ·001 of its amount at the height of 56 kilometres, or 35 miles.

Hypothesis B. On this hypothesis the expression for the density assumes the very simple form

$$\rho=\rho_1\left(1-\tfrac{1}{2}\frac{h}{h_1}\right). \qquad \text{(12)}$$

Although, as already pointed out, this hypothesis fixes the limit of the atmosphere at too low a point, it is, so far as the effect on refraction is concerned, markedly nearer the truth than that of constant temperature. The actual truth lies between the two but nearer to *B* than to *A*.

C. Bessel's Hypothesis. Bessel, disregarding any law based on temperature, proposed a law of density of the same general form with the exponential one of constant temperature, the exponent being multiplied by a factor less than unity. Although, in the most general form of its statement, it was implied that the factor might vary with the height, in practical use the constant factor

$$k=0{\cdot}9649,$$

was used. With this factor the expression for the density assumes the form

$$\frac{\rho}{\rho_1}=e^{-k\frac{h}{h_1}}. \qquad \text{(13)}$$

On this law was based the tables of refraction published in the *Tabulae Regiomontanae*, which have been in extensive use even up to the present time.

There is no law of variation of temperature which will correspond to this law of density. The fact is that, as a law, it is not possible. Every possible law of variation must make the pressure of the entire atmosphere equal to the pressure at the base. In order that this result may follow, it is necessary that the definite integral which expresses the entire weight of the atmosphere shall give the basic pressure. This requirement is expressed by the condition

$$\int_0^\infty \rho\, dh=\text{pressure at base}=h_1\rho_1.$$

But, by integrating the equation (13), it is seen that the pressure at the base of the atmosphere becomes, on Bessel's hypothesis.

$$p_1 = \frac{h_1 \rho_1}{k},$$

which is, therefore, too great by between three and four per cent. of its entire amount.

D. Ivory's hypothesis of uniform diminution of temperature with altitude. Taking the hypothesis of diminution of temperature at a uniform rate with the altitude, and assuming the rate to be proportional to the temperature at the base, the expression of the temperature in terms of the altitude and of the temperature τ_1 at the base will be of the form

$$\tau = \tau_1(1 - \beta h),$$

β being a constant factor. The constant rate of diminution is then

$$\frac{d\tau}{dh} = -\beta\tau_1, \quad \text{.............................(14)}$$

and the equation becomes

$$\frac{d\rho}{\rho dh} = \frac{\beta\gamma\tau_1 - g}{\gamma\tau_1(1 - \beta h)}.$$

The integration of this equation gives

$$\log \rho = \log \rho_1 + \frac{g - \beta\gamma\tau_1}{\beta\gamma\tau_1} \log(1 - \beta h),$$

$\log \rho_1$ being the arbitrary constant, so taken that, for $h = 0$, $\rho = \rho_1$.

If we put $$y = \frac{g - \beta\gamma\tau_1}{\beta\gamma\tau_1} = \frac{1}{\beta h_1} - 1, \quad \text{.....................(15)}$$

this equation will give

$$\frac{\rho}{\rho_1} = (1 - \beta h)^y.$$

One result of this law is that the atmosphere would terminate at the altitude for which $\beta h = 1$. If we put h_0 for this altitude, the preceding expression may be written

$$\frac{\rho}{\rho_1} = \left(1 - \frac{h}{h_0}\right)^y. \quad \text{...........................(16)}$$

The general average result of balloon observations is a rate of diminution which would place the absolute zero at the height of about 50 km. The exponent y may be written, putting $g=1$,

$$y+1=\frac{h_0}{h_1}=\frac{h_0}{\gamma\tau_1}. \quad \text{.........................(17)}$$

This would give $y=5$ for a surface temperature of 12° C., but, the exponent being a function of the temperature, will in general be a fractional quantity. For 0° C., we have, very nearly,

$$y=5{\cdot}25.$$

Hyp. E. Diminution of temperature in constant geometrical progression with altitude. If we put

β, the rate of proportional decrease.

The expression for the rate of diminution of the temperature in terms of the altitude will be

$$\frac{d\tau}{dh}=-\beta\tau, \quad \text{.........................(18)}$$

τ being the temperature at any point.

Integration gives $$\log\frac{\tau_1}{\tau}=\beta h,$$

or $$\frac{\tau}{\tau_1}=e^{\beta h}, \quad \text{.........................(19)}$$

as a relation between the temperature of any two points on the same vertical line at altitudes differing by h.

Accepting this law of temperature, the equation (7) shows the corresponding law for the density. Substituting (18) in (7), we have

$$\frac{1}{\rho}\frac{d\rho}{dh}=\beta-\frac{g}{\gamma\tau_1}e^{\beta h},$$

τ_1 being the temperature for $h=0$.

Integration gives $$\log\rho=\beta h+C-\frac{g}{\beta\gamma\tau_1}e^{\beta h},$$

C being the arbitrary constant of integration. Putting ρ_1, the temperature for $h=0$, we have

$$C=\log\rho_1+\frac{g}{\beta\gamma\tau_1}.$$

Thus the expression for the density becomes

$$\log\frac{\rho}{\rho_1}=\beta h-\frac{g}{\beta\gamma\tau_1}(e^{\beta h}-1). \quad \text{..................(20)}$$

Representing this quantity by $-v$, we have

$$\frac{\rho}{\rho_1}=e^{-v}, \quad \text{.................................(21)}$$

where

$$v=\frac{g}{\beta\gamma\tau_1}(e^{\beta h}-1)-\beta h. \quad \text{............(22)}$$

This expression is too complex for convenient use, unless the process of integrating the differential of the refraction is made largely numerical.

As it may be of interest to compare the density at different heights on the two hypotheses, the following table has been prepared. In each case there are two constants to which values at pleasure may be assigned. One is the temperature t_1 at the station; the other the rate of diminution with altitude.

101. Comparison of densities of the air at different heights on hypotheses *D* and *E*.

In *D* as used, the limit of the atmosphere is taken to be 48 km., and the rate of diminution of temperature to be $\tau_1 \div 48$, which, for 0° C., is nearly 5°·5 per km. On hypothesis *E* the rate of diminution is taken to be $\tau \div 45$ throughout. This is 6° C., very nearly, at 0° C.

	Hyp. *D*.			Hyp. *E*.			*E* - *D*.
	$\tau_1=261°·8$	$\tau_1=272°·6$	$\tau_1=284°·6$	$\tau_1=246°·5$	$\tau_1=271°·5$	$\tau_1=296°·5$	271°·5
h	$t_1=-9°·7$	$t_1=+1°·1$	$t_1=+13°·1$	$t_1=-25°$	$t_1=0°$	$t_1=+25°$	$t_1=0°$
0	1·00000	1·00000	1·00000	1·00000	1·00000	1·00000	0
5	0·56130	0·57694	0·59310	0·53426	0·57373	0·60347	- ·00167
10	0·29335	0·31095	0·32420	0·25489	0·30343	0·33815	- ·00603
15	0·13982	0·15368	0·16870	0·11658	0·14026	0·17528	- ·01203
20	0·05904	0·06754	0·07729	0·04635	0·06508	0·08280	- ·00158
25	0·02102	0·02526	0·03021	0·01637	0·02657	0·03538	+ ·00175
30	0·00580	0·00742	0·00948	0·00504	0·00896	0·01347	+ ·00172
35	0·00105	0·00146	0·00200	0·00134	0·00273	0·00452	+ ·00132
40	0·00008	0·00013	0·00020	0·00030	0·00071	0·00129	+ ·00059

Section II. Elementary Exposition of Atmospheric Refraction.

102. It is a general law of optics that the course followed by a ray of light is along the same line or curve for the two opposite directions in which the light may move. It follows that the course of a ray of light from a celestial object may equally well be studied as that of a ray passing from the observer to the object. Since, in practical astronomy, the given quantity is commonly the direction of the ray when it reaches the observer, this reverse direction is generally the simplest to consider. But the differential equations are the same in both cases.

The apparent zenith distance of a celestial object is that of the ray when it reaches the observer. The true zenith distance is the angle which the ray makes with the vertical of the observer's station before it enters the atmosphere. We readily see that the curvature of the ray is always concave toward the earth, so that the effect of refraction is always to make the apparent less than the true zenith distance. Hence the correction to the Z.D. for refraction is always positive.

The atmospheric strata, being always perpendicular to the direction of the vertical at the point, are separated by curved surfaces, of which the curvature is determined by that of the geoid. Practically they may be treated as spherical above any one region of the earth. If the zenith distance is small, the curvature is so slight that the refraction will then be nearly the same as if the surfaces of the strata were planes. A close approximation to the refraction at moderate zenith distances may, therefore, be obtained on this hypothesis.

103. Refraction at small zenith distances.

The following theorem forms the starting point in the subject:

Regarding the atmospheric strata as plane and parallel, the total amount of refraction is independent of the law of diminution of the refractive power with height, and depends solely upon the refractive power of the air at the surface of the earth.

To show this we put:

$\mu_0, \mu_1, \mu_2, \ldots \mu_n$, the indices of refraction of any number of successive atmospheric strata, from the outer limit of the atmosphere to the earth, which values start from $\mu_0 = 1$ as their beginning.

$z_0, z_1, z_2, \ldots z_n$, the angles of incidence at which the ray enters the successive strata, which are the same as those of refraction at which it leaves the bottom of each stratum above.

Then, by the law of refraction, we have

$$\begin{aligned} \sin z_1 : \sin z_0 &= \mu_0 : \mu_1 \\ \sin z_2 : \sin z_1 &= \mu_1 : \mu_2 \\ &\vdots \\ \sin z_n : \sin z_{n-1} &= \mu_{n-1} : \mu_n. \end{aligned}$$

These equations being multiplied give

$$\sin z_0 : \sin z_n = \mu_n : \mu_0.$$

Including all the strata of the atmosphere, we have

z_0, the true zenith distance;

z_n, the apparent zenith distance;

$\mu_0 = 1$;

μ_n, the index of refraction at the station.

Thus, in the case of plane strata, the relation between the true and the apparent zenith distance is given by the equation

$$\sin z_0 = \mu_n \sin z_n. \qquad (23)$$

This equation expresses the direction of the ray which, striking the atmosphere at the angle z_0, is refracted so as to reach the earth at the angle z_n. Being independent of all the values of μ except the last, it proves the theorem.

From this equation we may derive an approximate expression for the refraction when quantities of the second order as to its amount are neglected. Let us put:

R, the total refraction.

We then have $z_0 = z_n + R$,

and developing $\sin z_0$ to quantities of the first order in R,

$$\sin z_0 = \sin z_n + R \cos z_n. \qquad (24)$$

Equating (23) and (24), we have

$$\left.\begin{aligned} R\cos z_n &= (\mu_n - 1)\sin z_n \\ R &= (\mu_n - 1)\tan z_n \end{aligned}\right\} \quad \text{.....................(25)}$$

We, therefore, have the theorem:

At small zenith distances the refraction is proportional to the tangent of the zenith distance.

Let us form an idea of the error of this theorem for a zenith distance of 45°. From what has already been shown as to the density of the atmosphere, the larger portion of the refraction takes place at altitudes below 15 kilometres. The supposed ray reaches this height at the horizontal distance of 15 km. from the station. The curvature of the strata within this distance is approximately 8′. The numerical value of $\mu - 1$ is, as we shall see, not far from 0·0003. The change in the refraction produced by a change of 8′ in the value of z is of the order of magnitude $8' \times 0{\cdot}0003$; or about 0″·15. It follows that the correction required by the law of tangents is small even at a zenith distance as great as 45°.

The value of $\mu - 1$ at ordinary temperatures, when reduced to arc, ranges between 57″ and 60″. The number of degrees in the unit radius being 57·7, it follows from the equation (25) that, in the immediate neighbourhood of the zenith the refraction is approximately 1″ for each degree of zenith distance.

104. Let us now investigate the amount of the refraction, supposing the atmosphere to consist of an indefinite number of infinitely thin curved strata, each of the thickness dh.

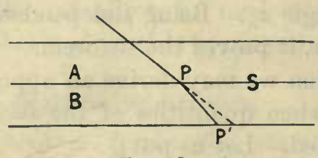

FIG. 18.

Let P be the point at which the ray intersects the bounding surface S between any two consecutive strata, A and B;

$\zeta + d\zeta$, the angle of incidence at P;

ζ, the angle of refraction after passing P.

$d\zeta$ is then the change of direction which the ray undergoes from one stratum into the next, when referred to any fixed direction. Putting μ and $\mu+d\mu$ for the indices of the two strata, we shall have

$$(\mu+d\mu)\sin(\zeta+d\zeta)=\mu\sin\zeta,$$

whence $$d\zeta=-\frac{d\mu}{\mu}\tan\zeta. \qquad (26)$$

When the ray reaches the lower surface of stratum B at a point which we shall call P', it strikes it at an angle of incidence $\zeta+\delta\zeta_1$, $\delta\zeta_1$ being the angle between the verticals at the points P and P'. It follows that the entire change of ζ from stratum to stratum is $d\zeta+\delta\zeta_1$; but the curvature of the ray being determined by reference to a fixed direction, is expressed by $d\zeta$ alone. The preceding equation is, therefore, a differential equation for the actual amount of the refraction.

105. Relation of density to refractive index.

To express (26) in terms of known quantities, we have next to express the index of refraction as a function of the density of the air. It has been a commonly accepted law of physics that the index of refraction of a gas is given as a function of its density by an equation of the form

$$\mu^2-1=2c\rho, \qquad (27)$$

c being a constant depending on the wave-length of the light and ρ the density of the gas.* The numerical value of the constant c may be determined either from the observed refraction of the stars, or from laboratory experiments. In practice the astronomer is obliged to leave out of consideration the variations in the refraction of light with its colour, and to base all his computations on the hypothesis that there is a certain mean or brightest region in the visible spectrum, the light of which he alone observes. There is an unavoidable indetermination as to the choice of a particular ray of the spectrum for this purpose,

* Recent experimental investigations make it probable that the yet simpler law $\mu=1+c\rho$ may be as near or nearer to the truth; but, for astronomical purposes, the difference between the results of the two formulae is unimportant.

added to which the differences between the colours of the different stars must involve differences in the refraction of their light. It is, however, remarkable that the most careful observations made up to the present time do not show any differences arising from this cause at moderate altitudes.

But this cause assumes another form near the horizon, where its effect must be sensible. Here the blue and even the green rays are nearly all absorbed by the atmosphere, leaving the visible body to be represented by rays from the lower part of the spectrum. The refraction must then be somewhat less than it would be as determined by any law which assumed the index of refraction to be the same at all altitudes.

Owing to this indetermination it is better to determine the index of refraction, or the value of c, from astronomical observations than from laboratory measures. The two are, however, in so good agreement that it is indifferent which we accept as the numerical basis of a theory.

Added to the source of indetermination just mentioned, we have the fact that the best investigations of the actual refraction suffered by the stars show a range of the thousandth part. In fact, Bessel's refraction tables, which have not yet gone wholly out of use, are based on an index of refraction greater by 0·003 of its amount than that of the Poulkova tables, which were constructed from the most refined observations. Yet the general consensus of recent observations is toward the view that the constant of the latter tables still requires a diminution of perhaps its thousandth part.

In the numerical investigations of the present chapter the adopted value of c is

$$c = 0{\cdot}226\,07,$$

a value which agrees closely with the Poulkova tables, and with the best laboratory measures of the refractive power of light near the ray D.

106. Form in which the refraction is expressed.

By differentiating (27), we find

$$\mu d\mu = c d\rho,$$

$$\mu^2 = 1 + 2c\rho.$$

These give, by division,

$$\frac{d\mu}{\mu}=\frac{c\,d\rho}{1+2c\rho}.$$

The density of the air is a minute fraction ranging from 0 to ·0012 or ·0013, the usual value of ρ at the earth's surface. We put ρ_1, the value of ρ at the point of observation. As ρ is a small factor, ranging from the value 0 at the upper limit of the atmosphere to ρ_1 at the station, we may, without appreciable error, replace the divisor $1+2c\rho$, which ranges between 1 and 1·0006, by its mean value, $1+c\rho_1$, which will give

$$\frac{d\mu}{\mu}=\frac{c\,d\rho}{1+c\rho_1}. \qquad (28)$$

Using the notation

$$\mathrm{c}=\frac{c}{1+c\rho_1}, \qquad (29)$$

and putting R for the amount of the refraction, (26) will give for its differential

$$dR=-d\zeta=\mathrm{c}\tan\zeta\,d\rho. \qquad (30)$$

This equation is rigorous. Conceive that we integrate it through the course of the ray from the station of the observer, where we have

$$\zeta=\text{apparent zenith distance}\equiv z,$$
$$\rho=\rho_1,$$

to the limit of the atmosphere where

$$\rho=0.$$

The total refraction will then be equal to the definite integral

$$\mathrm{c}\int_0^{\rho_1}\tan\zeta\,d\rho. \qquad (31)$$

Since ζ does not differ greatly from z at moderate zenith distances, it follows that a first approximation to the refraction in the region around the zenith will be derived by integrating as if $\tan\zeta$ had the constant value z, thus giving

$$R=\mathrm{c}\rho_1\tan z,$$

a result which, in principle, is equivalent to that expressed by (25). It follows that if we determine a factor m by the condition that the refraction shall be given rigorously in the form

$$R=\mathrm{mc}\rho_1\tan z, \qquad (32)$$

this factor will differ little from unity at moderate zenith distances. Its investigation being somewhat more intricate than is appropriate to the present section, is deferred to the following one. At present we shall show how, m being taken as known, the refraction is practically determined.

107. Practical determination of the refraction.

The density of the air at the station, or ρ_1, being the unknown factor in (32), we have to begin by showing the practical method of determining it at the moment of observation, and of bringing it into the theory. Its value is determined primarily by Eq. (2), § 94,

$$\rho = \frac{p}{\gamma\tau}.$$

Here γ is an absolute constant; ρ can, therefore, be immediately determined when the pressure p and the temperature τ are known. These two quantities are given by the readings of the barometer and thermometer. If the thermometer reads t degrees centigrade, the expression for τ is

$$\tau = 271{\cdot}5 + t. \qquad \text{(33)}$$

If the scale is that of Fahrenheit, the expression is

$$\tau = 271{\cdot}5 + \tfrac{5}{9}(t - 32).$$

The value of τ being found, we have shown in § 95, Eq. (9), that the value of ρ is given by the equation

$$\rho = \frac{0{\cdot}351\,11bg}{\tau}, \qquad \text{(34)}$$

where b is the ratio of the observed to the standard height of the barometer, the former being corrected for the temperature of the mercury in the barometer, while g is the ratio of gravity at the place to gravity at Paris.

Let us put

B, the observed reading of the barometer;

B_0, the standard height, 760 mm.;

κ, the coefficient of cubical expansion of mercury for 1°;

t', the temperature of the mercury above 0° C.

In strictness we should take for κ the excess of the cubical expansion of mercury over the cubical expansion of the tube of the barometer. But the latter is so small that it is neglected in practice, although there is no difficulty in the introduction of its approximate value for the special substance of which the tube is composed.

With these data the value of b will be given by the equation

$$b=\frac{B}{(1+\kappa t')B_0}. \qquad\qquad (35)$$

Introducing these various quantities, and putting

G, the force of gravity at the place, in any measure whatever;

G_0, the intensity of gravity at Paris, in the same measure;

the expression for ρ_1 is given in terms of known and observed quantities by the equation

$$\rho_1=\frac{0{\cdot}351\,11}{\tau}\cdot\frac{G}{G_0}\cdot\frac{B}{(1+\kappa t')B_0}. \qquad\qquad (36)$$

Substituting this value of ρ_1 in (32) the refraction will be expressed as a product of the five factors:

$$\text{cm}\tan z;\quad \frac{0{\cdot}351\,11}{\tau};\quad \frac{B}{B_0};\quad \frac{1}{1+\kappa t'};\quad \frac{G}{G_0}. \qquad\qquad (36a)$$

The logarithms of the first four of these factors are tabulated in different refraction tables used by the observatories. The factor depending on gravity has heretofore been very generally neglected, but should always be introduced if the fundamental constant c is not determined at the observatory itself. The first term is tabulated as a function of z, the apparent zenith distance; the others as functions of the actual reading of the external thermometer and barometer and the attached thermometer, which gives the temperature of the mercury. As to the special scale of temperature and of barometer height to be used as the argument of the tables, it need only be remarked that attention should be paid to see that the particular scale for which the tables are constructed is that to which the instruments are graduated.

The factor m, depending as it does on the curvature of the strata, differs little from unity in the neighbourhood of the zenith. A little consideration will make it evident that m must diminish as the zenith distance increases, because the angle between the course of the ray and the vertical at any point on the ray constantly diminishes with increasing height owing to the curvature of the strata. Since $\tan z$ increases without limit as the horizon is approached, while the refraction remains finite, the factor m must vanish at that point, and its logarithm must become infinite. The latter cannot, therefore, be advantageously used near the horizon.

The determination of m is necessary to the completeness of the theory. But as its value depends upon the law of diminution of density with increasing height, which, as has been seen in the preceding section, is very largely hypothetical, there can be no easily defined theory for the determination of m. As, on the hypotheses which come nearest to the truth, the developments necessary to determine this factor become intricate, the discussion of the subject is deferred to the following section.

108. Curvature of a refracted ray.

Let us now investigate the radius of curvature of the refracted ray when near the surface of the earth. We put

r, the radius of curvature;
s, the length measured along the ray.

The radius of curvature is given by the differential equation

$$\frac{1}{r}=\frac{d\zeta}{ds},$$

ds being the element of length of the ray.

The algebraic sign which we assign to $d\zeta$ is indifferent; we shall therefore always regard both $d\zeta$ and R as positive or signless quantities.

The value of $d\zeta$ is given by (26). Substituting for $d\mu:\mu$ its value (28), we have

$$d\zeta=\frac{c\,d\rho}{1+c\rho_1}\tan\zeta.$$

The product $c\rho_1$ in the denominator is so small that for our immediate purpose it may be disregarded. Substituting for $d\rho$ its value (7), we have

$$d\zeta=c\rho\left(\frac{g}{\gamma\tau}+\frac{1}{\tau}\frac{d\tau}{dh}\right)\tan\zeta\,dh.$$

The element of the length of the ray contained in the stratum is

$$ds = dh \sec \zeta.$$

The quotient of these equations give

$$\frac{d\zeta}{ds} = \frac{1}{r} = c\rho\left(\frac{1}{h_1} + \frac{1}{\tau}\frac{d\tau}{dh}\right)\sin z, \quad \text{.................}(37)$$

where we introduce the reciprocal of the pressure-height for $\frac{g}{\gamma\tau}$.

This equation shows that the curvature of the ray varies directly as $\sin z$, and, therefore, has its maximum value when the ray is horizontal. Let us next compute the value of the curvature for this case. We shall begin by supposing the temperature to be uniform. The equation (37) then gives for the radius of curvature

$$r = \frac{h_1}{c\rho}. \quad \text{.............................}(38)$$

Taking the case of air at standard density, we have, from numbers already given,

$$c\rho_0 = 0{\cdot}000\,293,$$

and putting $h_1 = 8$ km.,

$$r = 27\,300 \text{ km.} = 4{\cdot}3 \text{ radii of the earth.}$$

We have already learned that, near the earth's surface, the rate of change of temperature with height is very variable. Taking $-6^\circ{\cdot}5$ C. per kilometre as a normal rate, and 10° C. as a normal temperature, we shall have

$$\left.\begin{aligned} \frac{1}{\tau}\frac{d\tau}{dh} &= 0{\cdot}0231, \\ c\rho &= 0{\cdot}000\,282\,1, \\ h_1 &= 8{\cdot}28 \text{ km.}, \\ r &= 34\,600 \text{ km.}, = 5{\cdot}44 \text{ radii of the earth.} \end{aligned}\right\} \text{.......}(39)$$

As during the day the rate of diminution is yet greater than $6^\circ{\cdot}5$ per km., we may regard the ordinary curvature of a nearly horizontal ray as $\frac{1}{6}$ that of the geoid. But, owing to the cause already mentioned, this number is subject to wide variations. Not unfrequently the temperature-gradient along the course of the ray is positive. This is nearly always the case within an observing room, and within the tube of a telescope, owing to the tendency of the warmer air to rise to the top of a room or to the highest part of a tube. In this case the curvature will be greater than the normal. This particular phase of the subject belongs to the field of practical and instrumental astronomy, and will not be further considered at present.

The same result follows when a body of warm air passes over the frozen surface of the Arctic seas. At a certain temperature-gradient the curvature of the ray may become equal to or greater than that of the ocean itself. Then there will be no limit to the distance at which objects may be seen except that arising from non-transparency of the air. It is easy to define the temperature-gradient at which this effect follows. We have only to insert for r in (37) the value of the earth's radius, put $\sin z = 1$, and thus determine the value of $\frac{d\tau}{dh}$ as an unknown quantity. We then have

$$\frac{d\tau}{dh} = \frac{\tau_{\prime}}{ac\rho} - \frac{1}{\gamma} = 117°, \qquad (40)$$

where a is the radius of curvature of the geoid.

This implies a diminution of a little more than 1° C. or nearly 2° F. for each 10 metres of height.

The contrary case arises when the surface of a flat plain is heated by the rays of the sun. If the temperature gradient then has, near the ground, a negative value exceeding 1° in 34 metres, the ray will be concave toward the earth. For negative gradients largely exceeding this limit the ground at a distance may not be visible at all, a ray the line of which would reach the ground at a small angle being bent upwards and thus showing to the observer only the sky, while a higher ray where the gradient is more nearly normal will pass to and show an elevated object at a distance. In cases of this sort with a positive and rapidly diminishing gradient, an inverted image of distant objects may be seen. It is to this action of the temperature-gradient that the varied phenomena of mirage are due.

109. Distance and dip of the sea horizon.

By the *sea horizon* is meant the apparent boundary of the surface of the ocean when viewed through a transparent atmosphere. The plane of the observer's horizon is necessarily above the ocean, the latter receding farther and farther below it at a greater and greater distance. Consequently, the sea horizon is depressed below this plane. The angular amount of this depression is called the *dip of the horizon*.

In Fig. 19 let the arc HS be a section of the ocean surface through the centre of the earth C;

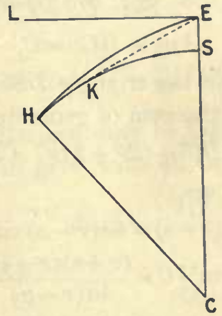

FIG. 19.

E, the position of the observer's eye;

S, the point in which the observer's vertical line intersects the surface of the ocean;

EL, a section of the horizontal plane through E;

EK, a tangent from the eye of the observer to the ocean surface at K.

Were there no refraction this tangent would be the course of a ray passing between K and the observer. Since LEC and CKE are both right angles, it follows that the angle LEK, or the dip of the horizon, would then equal ECK. That is to say, the dip, expressed in minutes, would be equal to the distance of the horizon in nautical miles. But, owing to the effect of refraction, the actual ray EH is concave to the surface of the ocean; consequently it is tangent to the latter at a point more distant than K. Let H be this point. We see from the figure that the actual dip is less than the geometric dip, and the actual distance of the sea horizon greater than the distance of the geometrical horizon.

To investigate the actual distance and dip, we remark that the height of the observer's eye above the ocean SE is practically so small that we may treat the ray as horizontal and disregard the effects of the height, thus supposing the curvature of the ray uniform. Let a point D, not shown in the figure, be the centre of curvature of the ray HE, which lies on the line HC produced. In the triangle CDS we therefore have

Side $DC = r - a$, r being the radius of the curvature of the ray. We also put

$$\text{Side } DS = s,$$
$$\text{Angle } HCS = C,$$

so that the angle at C of the triangle DSC is $180° - C$.

From a well-known theorem of geometry we have

$$DS^2 = DC^2 + CS^2 + 2DC \,.\, CS \cos C,$$

which gives the equations

$$\left.\begin{aligned} (r+s)(r-s) &= 4a(r-a)\sin^2\tfrac{1}{2}C \\ \sin^2\tfrac{1}{2}C &= \frac{(r+s)(r-s)}{4a(r-a)} \end{aligned}\right\} \quad \text{.................(41)}$$

It will be seen that C is the distance of the sea horizon in arc at the earth's centre. This, when expressed in minutes, corresponds to nautical miles. Let D be the distance of the horizon expressed in this way; then, owing to the minuteness of C, we may put

$$D = 6876 \sin \tfrac{1}{2} C,$$

the number 6876 being twice the number of minutes in the radian.

Owing to the minuteness of the height HE of the observer's eye above the ocean, and of the angle C, we may treat these quantities as infinitesimals. Putting

h, the height of the eye,

we shall then have, with all required precision,

$$r - s = h,$$
$$r + s = 2r,$$
$$\sin C = \sqrt{\frac{2rh}{a(r-a)}}.$$

With these various substitutions we find that the distance D is given in nautical miles by the equation

$$D = 3434\sqrt{\frac{2rh}{a(r-a)}}. \quad \text{.....................(42)}$$

The value of D therefore depends upon the radius of curvature r of the ray, which again is a function of the temperature-gradient. We have already shown how to express r as a function of this gradient. For all ordinary practical purposes we may suppose

$$r = 6a.$$

Then, taking the metre as the unit of length, we have $a = 637 \times 10^4$ and

$$D = 3438\sqrt{\frac{12h}{5a}} = 2'{\cdot}11\sqrt{h}. \quad \text{.....................(43)}$$

For the dip of the horizon may be taken the angle S of the triangle CSD, which will give, with all required precision,

$$\text{Dip} = \frac{r-a}{r}\sin C = \sqrt{\frac{2(r-a)h}{ar}}.$$

When $r = 6a$ this gives

$$\text{Dip in minutes} = 3438\sqrt{\frac{5h}{3a}} = 1'{\cdot}76\sqrt{h}.$$

We conclude that the distance of the sea horizon in nautical miles is about $\frac{1}{7}$ greater than the square root of the height of the observer's eye in feet. From the deck of an ocean liner, on which the eye is about 20 feet above the sea, the distance is not far from 5 miles. The line of sight being tangent to the ocean, its height above the ocean beyond the horizon is given by the reversal of the formula for the distance. From this it may be concluded that, at a distance of 10 miles, only the bridge of another ship will be visible, and that 15 or 16 miles is about the greatest distance at which the upper parts of the smokestack can be seen.

Section III. General Investigation of Astronomical Refraction.

110. In the preceding section the results of refraction have been treated rigorously only as regards the fundamental principles and on the hypothesis that the equiponderant strata of the atmosphere are plane. We have now to consider more rigorously than before the effect of the curvature of the strata upon the refraction. In strictness the equiponderant surfaces should be regarded as ellipsoidal, corresponding approximately to the figure of the geoid. It follows that in the region of the two poles, the curvature of the surfaces may be regarded as spherical, while, at the equator, the differences in the different azimuths reaches its maximum. The question whether this

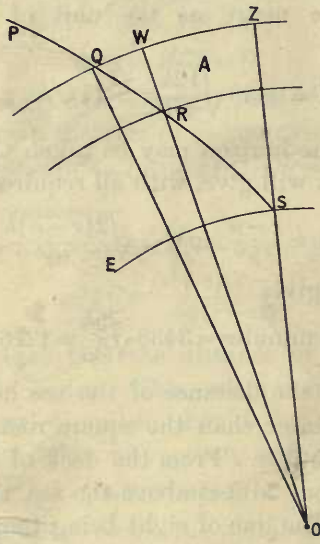

FIG. 20.

difference of curvature in different azimuths and different latitudes appreciably affects the refraction can only be determined after the formulae for the amount of refraction have been developed and reduced to numbers. It may, however, be remarked that, since the observations for which a precise value of the refraction is necessary are almost exclusively those made in the meridian, the main question will be whether the differences in the curvature of the meridian in different latitudes have a sensible effect.

In Fig. 20, let

O be the centre of curvature of the geoid in that plane passing through the station with reference to which the refraction is to be expressed;

S, the station of observation;

ES, the level surface of the station.

Ordinarily the station will be situated near the surface of the geoid, so that *ES* may be regarded as representing this surface, but in theory we may regard *S* as at any elevation above it.

Z, the zenith of the station;

ZQ, the upper surface of a thin stratum *A* within which we regard the index of refraction as constant;

PQRS, the course of a ray of light undergoing refraction at the points *Q* and *R* on the upper and lower surfaces of the stratum *A*, and ultimately reaching *S*;

OQ, *ORW*, vertical lines from the centre of curvature through the points *Q* and *R*.

We assume that the level surfaces are all concentric and therefore equidistant.

In order to express the refraction we begin with the conception of the atmosphere as formed by an indefinite number of successive strata, each of uniform density. Then by passing to the limit in which the number of strata becomes infinite and the differences of density in two consecutive strata infinitesimal, we have the case of the continuously varying density of the atmosphere.

Let us now study the refraction which a ray suffers in passing from *P* through *Q* and *R*. For this purpose we put

μ, the index of refraction for the stratum next above *A* whose lower surface is *QWZ*;

μ', the index for the stratum *A* itself;

ζ, the angle of incidence at which the ray falls at *Q* upon the upper surface of the stratum *A*;

ζ', the angle of refraction *OQR* at which it enters the stratum *A*;

ζ_1, the angle of incidence *QRW* at which it strikes the lower surface of *A*;

r, r_1, the radii of curvature of the upper and lower surfaces of the stratum *A*.

The law of refraction gives the equation

$$\mu' \sin \zeta' = \mu \sin \zeta.$$

In the triangle ROQ we have

$$\text{Angle } Q = \zeta',$$
$$\text{Angle } R = 180° - \zeta_1,$$

whence

$$r \sin \zeta' = r_1 \sin \zeta_1.$$

Eliminating ζ' from these equations, we have

$$r_1 \mu' \sin \zeta_1 = r\mu \sin \zeta.$$

It follows that the product $r\mu \sin \zeta$ has the same value in every two consecutive strata, and is therefore constant for the whole course of the ray through the atmosphere. Its value may, therefore, be derived from its value at the base, to express which we put

a, the radius of curvature of the geoid at the station;
μ_1, the index of refraction at that point;
z, the apparent zenith distance of the body.

The value of the constant in question will thus be

$$a\mu_1 \sin z \equiv C.$$

Passing now to the actual case in which the increase of density is a continuously varying quantity, the successive strata become infinitely thin, and the angle ζ becomes that which the ray, at each point of its course, makes with the vertical line at that point. We therefore have the equation

$$r\mu \sin \zeta = a\mu_1 \sin z. \quad \text{.........................(1)}$$

The second member of this equation being supposed to contain only given and known quantities, the equation expresses a relation between r, μ, and ζ at all points on the course of any one ray.

The refraction being the total change which the direction of the ray undergoes from the point of observation to the outer limit of the atmosphere, it follows that the differential of the refraction is, at each point, the infinitesimal curvature of the ray. This is the same as that part of the change in the angle ζ which arises from the change in the value of the index μ from one point to another. We therefore have the differential of the refraction by writing dR for $d\zeta$ in equation (26), Sect. II.,

$$dR = -\frac{d\mu}{\mu}\tan\zeta. \quad\text{..........................(2)}$$

In this equation we substitute for $\tan\zeta$ its value derived from (1),

$$\tan\zeta = \frac{\mu_1 a \sin z}{\sqrt{\{\mu^2 r^2 - \mu_1{}^2 a^2 \sin^2 z\}}}.$$

The differential equation (2) of the refraction thus becomes, dropping the negative sign as indifferent,

$$dR = \frac{d\mu}{\mu} \cdot \frac{\sin z}{\sqrt{\left\{\frac{\mu^2 r^2}{\mu_1{}^2 a^2} - \sin^2 z\right\}}}. \quad\text{..................(3)}$$

This is a rigorous equation which, being integrated as to the variable μ from the outer limit of the atmosphere to the point of observation, or the reverse, gives the total refraction.

111. Transformation of the differential equation.

In the integration the apparent zenith distance z, and the radius of curvature a, are constants, while μ and r are variables. To reduce the expression to an integrable form, a number of transformations are required.

Our first two transformations will consist in replacing $\frac{r}{a}$ by its expression in terms of the height, and expressing $\frac{\mu}{\mu_1}$ in terms of the density of the air. Putting, as before, h for the height, we have

$$\frac{r}{a} = \frac{a+h}{a} = 1 + \frac{h}{a}. \quad\text{........................(4)}$$

It will conduce to clearness to express h in terms of the pressure height h_1 as the unit of height. This we do by introducing the variable x,

$$x = \frac{h}{h_1}. \quad\text{..................................(5)}$$

We have found h_1 to be a function of the temperature τ_1 at the station defined by the equation

$$h_1 = \frac{\gamma\tau_1}{g}. \quad\text{..............................(6)}$$

We then have $$\frac{r}{a}=1+\frac{h_1}{a}x.$$

From the relation between the index of refraction and the density

$$\mu^2-1=2c\rho, \qquad (7)$$

we have $$\frac{\mu^2}{\mu_1^{\ 2}}=\frac{1+2c\rho}{1+2c\rho_1}=1-\frac{2c\rho_1\left(1-\frac{\rho}{\rho_1}\right)}{1+2c\rho_1}.$$

By introducing the constant α,

$$\alpha=\frac{c\rho_1}{1+2c\rho_1}, \qquad (8)$$

and the variable w, $$w=1-\frac{\rho}{\rho_1}, \qquad (9)$$

we have $$\frac{\mu^2}{\mu_1^{\ 2}}=1-2\alpha w.$$

Then, putting for brevity,

$$\nu=\frac{h_1}{a}=\frac{g\gamma\tau_1}{a}, \qquad (10)$$

which gives $$\frac{r}{a}=1+\nu x,$$

we have $$\frac{\mu^2}{\mu_1^{\ 2}}\frac{r^2}{a^2}=1+2u, \qquad (11)$$

where $$u=\nu x-\alpha w+\tfrac{1}{2}\nu^2x^2-\alpha\nu wx(2+\nu x). \qquad (12)$$

Making in the equation (3) the substitutions (11) and (28) of § 106, we have

$$dR=\frac{c}{1+c\rho_1}\tan z\,\frac{d\rho}{\sqrt{(1+2u\sec^2 z)}}. \qquad (12a)$$

Our next step is to substitute w for ρ as the variable. From (9),

$$d\rho=-\rho_1 dw. \qquad (13)$$

Substituting for $d\rho$ this value, we shall put

$$\mathrm{a}=\frac{c\rho_1}{1+c\rho_1}=\frac{\alpha}{1-\alpha}. \qquad (14)$$

With these substitutions the general equation for the refraction may be written in either of the forms,

$$R=\mathrm{a}\tan z\int_0^1\frac{dw}{\sqrt{1+2u\sec^2 z}}, \qquad (15a)$$

$$R=\mathrm{a}\sin z\int_0^1\frac{dw}{\sqrt{2u+\cos^2 z}}. \qquad (15b)$$

The first of these forms is most conveniently applicable, except within a few degrees of the horizon, when we are obliged to have recourse to the second. It was shown in Section II. that for small zenith distances the refraction is proportional to the tangent of the zenith distance, and that, therefore, except near the horizon, the refraction is usually expressed in the form

$$R = \mathrm{am} \tan z. \quad \text{.........................(16)}$$

We now see that the factor m will be given by the equation

$$\mathrm{m} = \int_0^1 \frac{dw}{\sqrt{1+2u\sec^2 z}}, \quad \text{.....................(17)}$$

the investigation of which next demands our attention.

112. The integration.

The great problem of astronomical refraction consists in the integration of the preceding equations. This problem offers no serious difficulty in the case of zenith distances to 80°, except that of deciding upon a law according to which the density of the air diminishes with the height. It will be well to preface a consideration of the problem by a review of its general nature, and of the forms in which it has to be attacked in different cases. First let us form an idea of the order of magnitude of the quantities with which we have to deal, especially of u.

From what we have seen of the law of density of the air, it follows that all the refraction with which we need concern ourselves takes place below the altitude of 60 kilometres, and that it is very small above 40 kilometres. For this altitude we have, approximately,

$$x = 5.$$

Also

$$\nu = \frac{h_1}{a} = \frac{8}{6360} = \frac{1}{800}.$$

Hence we always have

$$\nu x < 0{\cdot}01,$$

while values greater than 0·006 add very little to the refraction. w ranges between the limits 0 and 1, and as $\alpha < 0{\cdot}0003$, it follows that

$$\alpha w < 0{\cdot}0003.$$

These are the two largest terms of u, so that, whenever

$$\sec z < 7,$$

we have
$$2u \sec^2 z < 1,$$

and the denominator of (17) may be developed in powers of this quantity.

Regarding ν and α as quantities of the first order, the three last terms of u in (11) are of the second and third orders. Owing to the minuteness of ν and α these higher terms are of minor importance and their consideration will, therefore, be postponed. Dropping them from the value of u, the latter becomes

$$u = \nu x - \alpha w. \quad \text{.........................(18)}$$

The difficulty of the problem arises from the fact that the two terms of u must be expressed as a function of some one variable before the integration can be effected. Between these terms the relation is of a complex character. It will be profitable to apprehend the respective origins of the two terms of u. We recall that x may be defined as the altitude above the earth of any point of the refracted ray, expressed in terms of the pressure height as the unit. The factor ν arises from the rotundity of the earth, being the reciprocal of the earth's radius of curvature, which would vanish were there no curvature. The product νx may be described as due to the change of the angle between the ray and the vertical line at each point, so far as this change arises from the earth's rotundity. Considered as passing in the reverse direction from the observer outwards, the height of the ray at any point depends both on the curvature of the earth and on that of the ray itself. The term αw may be described as arising from the curvature of the ray.

Assuming the density of the air to be a given function of the height, the quantity w,

$$w = 1 - \frac{\rho}{\rho_1}$$

becomes a function of x, which, being substituted in u, enables the latter to be expressed as a function of x. Then, replacing the differential of w by that of x, the problem will become that of integrating with respect to x as the independent variable. This offers no difficulty in using the form (15a), but is not practicable when $\sec z$ is so large that the form (b) has to be used.

In Section I. five hypotheses have been set forth as to the relation between the density of the air and the height. These hypotheses lead to as many expressions for the relation between w and x. Of these hypotheses it may be said, in a general way, that the first three lead to forms which admit of being integrated by well-known methods; but that all three of them deviate in a definable way from the actual facts of the case. The remaining two, by adopting the proper factor of diminution of density with altitude, can probably be made to represent the facts as accurately as is necessary for the purpose of refraction.

113. Development of the refraction.

The preceding considerations suggest the separate consideration of the problem in its two forms. In one of these forms we use (15a) and develop in powers of u; in the other we have to consider the method of dealing with the integration when the required development in powers of u is impracticable. The limits of the zenith distance within which the first method is applicable will appear after the developments are effected.

The purpose of this method is the determination of m from the equation (17). The development of the denominator in powers of u by the binomial theorem gives

$$(1+2u\sec^2 z)^{-\frac{1}{2}} = 1 - u\sec^2 z + \tfrac{3}{2}u^2\sec^4 z \;\ldots, \qquad (19)$$

the coefficient of $(-1)^i u^i$ being

$$\frac{1.3.5\ldots(2i-1)}{1.2.3\ldots i} \equiv [i].$$

The first five coefficients, taken positively, are

$$\left.\begin{aligned} [1] &= 1 \\ [2] &= \tfrac{3}{2} \\ [3] &= \tfrac{5}{2} \\ [4] &= \tfrac{35}{8} \\ [5] &= \tfrac{63}{8} \end{aligned}\right\} \qquad (20)$$

If we express m in the form

$$\text{m} = 1 - m_1 \sec^2 z + m_2 \sec^4 z - \ldots, \quad \ldots\ldots\ldots\ldots\ldots(21)$$

the general value of m_i will be

$$m_i = [i]\int u^i dw. \quad \ldots\ldots\ldots\ldots\ldots(22)$$

Putting for u^n its value in terms of m_n, the definite integrals which enter into this expression are found by integrating the following forms between the limits 0 and 1:

$$\left.\begin{aligned}
u\,dw &= \nu x\,dw - \alpha w\,dw \\
u^2dw &= \nu^2x^2dw - 2\alpha\nu xw\,dw + \alpha^2w^2dw \\
u^3dw &= \nu^3x^3dw - 3\alpha\nu^2x^2w\,dw + 3\alpha^2\nu xw^2dw - \alpha^3w^3dw \\
u^4dw &= \nu^4x^4dw - 4\alpha\nu^3x^3w\,dw + 6\alpha^2\nu^2x^2w^2dw - \ldots \\
u^5dw &= \nu^5x^5dw - 5\alpha\nu^4x^4w\,dw + \ldots \\
&\vdots \qquad \vdots \qquad \vdots
\end{aligned}\right\} \ldots\ldots(23)$$

When dw is replaced by its value in terms of dx, the limits of integration will be 0 and the value of x corresponding to the height of the atmosphere, or of the absolute zero of temperature. When taken between these limits, the integrals admit of farther simplification. The last terms in each of the differentials being independent of x, are numerical constants multiplied into successive powers of α. They are, therefore, like the principal terms of the refraction, dependent only upon the density of the air at the point of observation. To reduce $\int x\,dw$, we substitute for dw its expression in terms of $d\rho$,

$$dw = -\frac{d\rho}{\rho_1}.$$

We thus have

$$\int x\,dw = -\frac{1}{\rho_1}\int x\,d\rho.$$

Integrating by parts,

$$\int x\,d\rho = \rho x - \int \rho\,dx.$$

At the lower limit of integration we have $x=0$, and at the upper limit $\rho=0$. Hence the product ρx vanishes at both limits, leaving as the integral

$$\int_0^\infty \rho\, dx = \frac{1}{h_1}\int_0^\infty \rho\, dh.$$

This integral expresses the total mass of the atmosphere contained in a vertical column of unit base. It is, therefore, independent of the law of density. We therefore have the remarkable theorem:

In the development of m *in powers of* sec z *the coefficient of* $\sec^2 z$ *is independent of the law of diminution of the density of the air with its height.*

It has already been pointed out that the Besselian law of density of the air gives a total mass greater than the actual mass as indicated by the pressure and temperature at the base. It follows that in this theory the coefficient of $\sec^2 z$ is too large in the same ratio. To determine the coefficient in question we note that the total mass of the column of air is the product of the density at its base into the pressure-height. We therefore have

$$\int_0^\infty \rho\, dh = \rho_1 h_1.$$

Making these successive substitutions we have

$$\int_0^\infty x\, dw = 1, \qquad (24)$$

and then

$$m_1 = \nu - \tfrac{1}{2}\alpha, \qquad (25)$$

for all laws of density.

114. Passing to the determination of the higher values of m_n, we see from (22) and (23) that the integrals required are all of the general form

$$\int_0^1 x^n w^i\, dw.$$

To investigate the value of this integral, we first substitute for w and dw their expressions in terms of ρ, thus obtaining

$$\left.\begin{aligned} w^i &= 1 - i\frac{\rho}{\rho_1} + \frac{i(i-1)}{1\,.\,2}\frac{\rho^2}{\rho_1^2} - \cdots \\ dw &= -\frac{d\rho}{\rho_1} \end{aligned}\right\} \qquad (26)$$

Thus we have

$$x^n w^i dw = -x^n \frac{d\rho}{\rho_1} + ix^n \frac{\rho\, d\rho}{\rho_1^2} - \frac{i(i-1)}{1\,.\,2} x^n \frac{\rho^2 d\rho}{\rho_1^3} + \dots .$$

These terms are to be integrated for w between the limits 1 and 0. The integrals to be found now become of the general form

$$\int_0^{\rho_1} x^n \rho^\kappa d\rho \equiv \rho_1^{\kappa+1} I_{n,\kappa}. \quad \dots\dots\dots\dots(27)$$

Interchanging the limits of integration as to ρ, we then have

$$\int_0^1 x^n w^i dw = I_{n,0} - iI_{n,1} + \frac{i(i-1)}{1\,.\,2} I_{n,2} - \dots . \quad \dots\dots\dots(28)$$

The values of $I_{n,\kappa}$ depend on the relation between the density and the height, for which we shall take the two fundamental hypotheses A and D.

115. Development on Newton's hypothesis.

Hypothesis A gives

$$\rho = \rho_1 e^{-x},$$
$$d\rho = -\rho_1 e^{-x} dx;$$
$$\therefore\ I_{n,\kappa} = \int_0^\infty e^{-(\kappa+1)x} x^n dx.$$

This is a well-known Eulerian integral, of which the value may be found by integrating by parts, leading to the result

$$I_{n,\kappa} = \frac{1\,.\,2\,.\,3 \dots n}{(\kappa+1)^{n+1}}. \quad \dots\dots\dots\dots(29)$$

Assigning to κ the successive values 0, 1, 2 ... (28) becomes

$$\int_0^1 x^n w^i dw = n!\left\{1 - \frac{i}{2^n\,.\,2!} + \frac{i(i-1)}{3^n\,.\,3!} - \frac{i(i-1)(i-2)}{4^n\,.\,4!} + \dots\right\}. \quad (30)$$

This general form, by assigning suitable values to n and i, gives the coefficients of $\nu^n \alpha^i$, which appear in the integration of (22), and thus lead to the following expressions:

$$\left.\begin{aligned} \int_0^1 u^2 dw &= 2\nu^2 - \tfrac{3}{2}\alpha\nu + \tfrac{1}{3}\alpha^2 \\ \int_0^1 u^3 dw &= 6\nu^3 - \tfrac{21}{4}\alpha\nu^2 + \tfrac{11}{6}\alpha^2\nu - \tfrac{1}{4}\alpha^3 \\ \int_0^1 u^4 dw &= 24\nu^4 - \tfrac{45}{2}\alpha\nu^3 + \dots \\ \int_0^1 u^5 dw &= 120\nu^5 - \tfrac{465}{4}\alpha\nu^4 + \dots \end{aligned}\right\} \quad \dots\dots\dots(31)$$

These being multiplied by the factors $[i]$ give the values of $m_2 \dots m_5$ on the Newtonian hypothesis.

116. Development on Ivory's hypothesis.

Here the relation between w and x is given by (16) and (17) of § 100. We recall that the height h_0 is that of the absolute zero, supposing the temperature to go on diminishing at a constant rate with increasing altitude, which it seems to do up to the highest point to which explorations extend. We put

$$\mathrm{v} = \frac{h_0}{h_1}.$$

Then (16) becomes $$\frac{\rho}{\rho_1} = \left(1 - \frac{x}{\mathrm{v}}\right)^{\mathrm{v}-1}. \qquad \text{(32)}$$

We now replace x by another variable y,

$$y = 1 - \frac{x}{\mathrm{v}}.$$

Then
$$\left.\begin{aligned} x &= \mathrm{v}(1-y) \\ \rho &= \rho_1 y^{\mathrm{v}-1} \\ dx &= -\mathrm{v}\,dy \\ w &= 1 - y^{\mathrm{v}-1} \\ d\rho &= (\mathrm{v}-1)\rho_1 y^{\mathrm{v}-2} dy \end{aligned}\right\} \qquad \text{(33)}$$

We then have for substitution in the first member of (27),

$$x^n \rho^\kappa d\rho = \mathrm{v}^n (\mathrm{v}-1) \rho_1^{\kappa+1} (1-y)^n y^{\kappa(\mathrm{v}-1)+\mathrm{v}-2} dy.$$

For $\rho = \rho_1$ we have $y = 1$ and for $\rho = 0$ $y = 0$. Substituting these expressions and these limits of integration in (27), we find

$$I_{n,\kappa} = \mathrm{v}^n(\mathrm{v}-1)\int_0^1 (1-y)^n y^{\kappa(\mathrm{v}-1)+\mathrm{v}-2} dy. \qquad \text{(34)}$$

This is a Eulerian integral which can be evaluated by successive integration by parts so as to reduce the exponent n step by step to 0. The development of this process belongs to the integral calculus. We shall, therefore, only state the general result.

For this purpose the Γ functions of Euler, or the Π functions of Gauss, come into play. The only difference between these functions is one of notation,

$$\Pi(n) = \Gamma(n+1).$$

The Π form of expression is most convenient for our use, because, when n is a positive integer,

$$\Pi(n) = 1.2.3 \ldots n = n!$$

Taking m and n as the exponents, the known general value of the integral is

$$\int_0^1 (1-y)^n y^m dy = \frac{\Pi(m)\Pi(n)}{\Pi(m+n+1)}, \quad \ldots\ldots\ldots\ldots(35)$$

which is easily computed when m and n are small positive integers.

Using this general form in (27), we have

$$I_{n,\kappa} = (\mathrm{v}-1)\mathrm{v}^n \frac{\Pi(n)\Pi\{\kappa(\mathrm{v}-1)+\mathrm{v}-2\}}{\Pi\{n+(\kappa+1)(\mathrm{v}-1)\}}. \quad \ldots\ldots\ldots(36)$$

Although, in the form in which we have stated the hypothesis, v is a function of the temperature, the rate of diminution is still so far doubtful that, practically, nothing is lost in the present state of our knowledge by using a constant value of v. We shall therefore put

$$\mathrm{v} = 6,$$

which amounts to supposing the absolute zero to be reached at 6 times the pressure height, whatever that may be, and the rate of diminution of τ with the height to be always 5°·6 per kilometre.

Then $$I_{n,\kappa} = 5 \cdot 6^n \frac{n!(5\kappa+4)!}{(n+5(\kappa+1))!}. \quad \ldots\ldots\ldots\ldots(37)$$

For the values 0, 1, and 2 of κ we have

$$I_{n,0} = 5 \cdot 6^n \frac{4!\, n!}{(n+5)!},$$

$$I_{n,1} = 5 \cdot 6^n \frac{9!\, n!}{(n+10)!},$$

$$I_{n,2} = 5 \cdot 6^n \frac{14!\, n!}{(n+15)!},$$

which are the only values we need for our present purpose.

By assigning to n the special values 2, 3, 4, ... and substitution in (28), with $i=0$, 1, and 2, we find

$$\int_0^1 x^2 dw = \tfrac{12}{7},$$

$$\int_0^1 xw\,dw = \tfrac{8}{11},$$

$$\int_0^1 x^3 dw = \tfrac{27}{7}.$$

Then, by substitution in (23),

$$\left.\begin{aligned}
\int_0^1 u^2 dw &= \tfrac{12}{7}\nu^2 - \tfrac{16}{11}\nu\alpha + \tfrac{1}{3}\alpha^2 \\
\int_0^1 u^3 dw &= \tfrac{27}{7}\nu^3 - \tfrac{375}{77}\nu^2\alpha + \tfrac{153}{88}\nu\alpha^2 - \tfrac{1}{4}\alpha^3 \\
\int_0^1 u^4 dw &= \tfrac{72}{7}\nu^4 - \tfrac{216}{43}\nu^3\alpha + \dots \\
\int_0^1 u^5 dw &= \tfrac{216}{7}\nu^5 - \text{etc.}
\end{aligned}\right\} \quad \dots\dots\dots\dots(38)$$

Then, from (22) we have, including the value of m_1 already found,

$$\left.\begin{aligned}
m_1 &= \nu - \tfrac{1}{2}\alpha \\
m_2 &= \tfrac{18}{7}\nu^2 - \tfrac{24}{11}\nu\alpha + \tfrac{1}{2}\alpha^2 \\
m_3 &= \tfrac{135}{14}\nu^3 - \tfrac{1875}{154}\nu^2\alpha + \tfrac{765}{176}\nu\alpha^2 - \tfrac{5}{8}\alpha^3 \\
m_4 &= 45\nu^4 - \tfrac{945}{43}\nu^3\alpha + \dots \\
m_5 &= 243\nu^5 - \text{etc.}
\end{aligned}\right\} \quad \dots\dots\dots\dots(39)$$

On comparing the values of the preceding integrals (38) with those derived on the Newtonian hypothesis (31) it will be seen that the coefficients and, therefore, the values of m_2, m_3, etc., on the latter hypothesis are constantly larger than those on Ivory's, which we regard as the most probable. This is quite in accordance with the fact, it being found that on the Newtonian hypothesis the refraction is too large as we approach the horizon, at which point it is about one-tenth greater than the true value.

117. Construction of tables of refraction.

The preceding completes the theory of the development in powers of sec z so far as the general expressions which determine the refraction are concerned. It is, however, necessary to show how the results of these expressions are put in the special form adopted in the tables of refraction as described in § 98.

The practical method now generally adopted of constructing such tables is due to Bessel. The logarithms of the four principal factors are tabulated in the following way:

Firstly, standard values of the temperature and pressure are adopted, and for these special values a table giving the logarithm of the refraction as a function of the apparent zenith distance is computed. We shall take as standards:

$$\left.\begin{array}{ll}\text{Temperature, } 50^\circ \text{ F.} & (\tau_{1,0}=281^\circ\!\cdot\!5) \\ \text{Pressure, 30 inches} & (B_1=762 \text{ mm.})\end{array}\right\} \qquad (40)$$

These are near the mean temperature and pressure at the active observatories, an approximation to which is desirable in choosing the standard temperature.

By substituting the preceding values of τ and B in (36) of § 107, we shall have a standard density $\rho_{1,0}$ of the air. Putting, for the time being, $G=G_0$, and taking an arbitrary standard temperature t_1' as that of the mercury in the barometer when it has the standard height $B_1=762$ mm., the standard density will become

$$\rho_{1,0}=\frac{0\cdot3511}{281\cdot5}\cdot\frac{762}{760(1+\kappa t_1')}. \qquad (41)$$

If, as is usual, we take 0° C. as the standard temperature for the mercury, we shall have $t_1'=0$, and the standard density will be

$$\rho_{1,0}=0\cdot001\,250\,5.$$

With this value of ρ_1 and $c=0\cdot226\,07$ (§ 95) we find from (14),

$$\mathrm{a}=0\cdot000\,282\,63=58''\!\cdot\!297. \qquad (42)$$

The expression (16) for the standard refraction thus becomes

$$R_0=58''\!\cdot\!297\,\mathrm{m}_0\tan z, \qquad (43)$$

where m_0 is the value of m for standard τ and B.

The general value of m is given by (21), where we are to substitute the values of the coefficients from (39). The latter contain ν in (10), to compute which we require the radius of curvature a of the geoid. This ranges between

$$\log a=6\cdot8017 \text{ at the equator}$$

and $$\log a=6\cdot8061 \text{ at the poles.}$$

For the value of g, the ratio of gravity at the place to that at Paris, we have

$$\log g = -0{\cdot}0013 \text{ at the equator}$$

and

$$\log g = +0{\cdot}0010 \text{ at the poles.}$$

From $\gamma = 29{\cdot}429$ m. (§ 87) and $\tau_1 = 281{\cdot}5$ we now find

$$\text{At the equator, } \nu = 0{\cdot}001\,309,$$

$$\text{At the poles, } \nu = 0{\cdot}001\,292.$$

The difference between these values is practically not important, because at low altitudes, where it might be sensible, the refraction is necessarily uncertain. The differences between the curvature of the atmospheric strata in different latitudes need not therefore be considered at present. We may use for all latitudes at standard τ,

$$\nu_0 = 0{\cdot}001\,30.$$

With this value of ν and the corresponding value of α,

$$\alpha_0 = 0{\cdot}000\,283,$$

we find the numerical values of $m_{1,0}$, $m_{2,0}$, etc., from (39),

$$m_{1,0} = 0{\cdot}001\,16,$$

$$m_{2,0} = 0{\cdot}000\,001\,4,$$

etc., etc.

Then from (21), $$m_0 = 1 - 0{\cdot}001\,16 \sec^2 z + \text{etc.}$$

At the zenith we have

$$m_0 = 0{\cdot}998\,84.*$$

* This expression for the refraction diverges from that usually derived in that the latter is developed in powers of $\tan z$ and the value of m becomes 1 at the zenith. The difference of form arises from the fact that the previous investigators have used instead of the symbol h employed in § 104 the quantity s, the ratio of the height h to $a+h$, the actual distance from the centre of curvature. The value of h thus appearing in the denominator complicates the theory and at the same time makes it less rigorous, because when we neglect the higher powers of s the factor of the refraction depending on curvature vanishes at the zenith. As a matter of fact, however, it does not so vanish, but converges toward the finite quantity found above, as can readily be seen by geometric construction. The difference is, however, little more than a matter of form and simplicity. It is easy to substitute the tangent for the secant in the preceding developments; but nothing would be gained by this course, except facilitating the comparison with former theories.

Thus, for small zenith distances, we have, under standard conditions,

$$R_0 = 58''{\cdot}230 \tan z.$$

This factor of $\tan z$ is what is properly called the constant of refraction. We have derived it by starting from the observed refractive index of air for the brightest part of the spectrum. But in practice it is derived from observations of zenith distances of the stars. The corresponding value of the Poulkova constant is

$$58''{\cdot}246.$$

Reduced to gravity at the latitude of Paris this would become

$$58''{\cdot}188,$$

a value slightly less than that just computed. Whatever the adopted value, the table of $\log R$ for standard conditions is readily computed.

118. The next step will be the tabulation of the logarithms of the factors for the deviations of the actual conditions from the standard ones. Returning once more to § 107 we see that, at any one station, ρ_1 contains three variable factors. Defining these factors as those by which we must multiply the standard density in order to form the actual density, they are:

1. Factor dependent on temperature of the external air,

$$T = \frac{\tau_{1,0}}{\tau_1} = \frac{281{\cdot}5}{271{\cdot}5 + \text{Temp. } C} \quad \text{.........................(44)}$$

2. Factor depending on barometer,

$$b = \frac{B}{B_1} = \frac{B}{762 \text{ mm.}} = \frac{B}{30 \text{ in.}}, \quad \text{.....................(45)}$$

according to the scale used on the barometer.

3. Factor dependent on the temperature of the mercury,

$$t'' = \frac{1}{1 + \kappa t'} \quad \text{..................................(46)}$$

The logarithms of these three factors are readily tabulated. They are to be multiplied by factors depending on the zenith distance and arising from taking account of the changes in the values of ν and α, and therefore in m_1, m_2, etc., arising from the

deviations from the standard conditions. To derive them we put, in (21),
$$\sigma = m_1 \sec^2 z - m_2 \sec^4 z + \ldots, \quad\ldots\ldots(47)$$
so that
$$\mathrm{m} = 1 - \sigma$$
and
$$\log \mathrm{m} = -M(\sigma - \tfrac{1}{2}\sigma^2 + \ldots), \quad\ldots\ldots(48)$$
M being the modulus of logarithms.

Putting σ_0 for the standard value of σ,
$$\sigma_0 = (\nu_0 - \tfrac{1}{2}\alpha_0)\sec^2 z = 0{\cdot}001\,16 \sec^2 z,$$
we shall have, when we drop the higher powers of σ,
$$\log \mathrm{m} - \log \mathrm{m}_0 = M(\sigma_0 - \sigma), \quad\ldots\ldots(49)$$
from which we may derive $\log \mathrm{m}$ when σ is known. Since the time of Bessel the universal practice has been to develop $\sigma_0 - \sigma$ in powers of $\log T$ and $\log b$, retaining only the first power. This is sufficiently accurate in practice except near the horizon, for which case Radau has developed an improved method. To show how Bessel's development is effected we need only the principal term of σ. Then (49) gives for the reduction of $\log \mathrm{m}$ from standard to actual conditions
$$\log \mathrm{m} - \log \mathrm{m}_0 = M(\nu_0 - \nu + \tfrac{1}{2}\alpha - \tfrac{1}{2}\alpha_0)\sec^2 z. \quad\ldots\ldots(50)$$

We now have to express ν and α in terms of T and b. Comparing (8) and (10) with (44) and (45) we see that, dropping insensible terms,
$$\nu = T^{-1}\nu_0,$$
$$\alpha = bT\alpha_0,$$
T and b being the factors (44) and (45). Thus we find
$$\left.\begin{aligned}\nu_0 - \nu &= \nu_0(1 - T^{-1})\\ \alpha - \alpha_0 &= \alpha_0(bT - 1)\end{aligned}\right\}. \quad\ldots\ldots(51)$$

T and b differ from 1 only by a fraction of which the average value within the range of temperatures at which observations are usually made, say -15° and $+30^\circ$, is less than 0·05. To quantities of the first order as to this difference we have
$$\left.\begin{aligned}M(1 - T^{-1}) &= \log T\\ M(bT - 1) &= \log T + \log b\end{aligned}\right\}, \quad\ldots\ldots(52)$$
and (50) takes the form
$$\log \mathrm{m} - \log \mathrm{m}_0 = \{(\nu_0 + \tfrac{1}{2}\alpha_0)\log T + \tfrac{1}{2}\alpha_0 \log b\}\sec^2 z.$$

The corresponding reductions of m_2, m_3, etc., may be developed by a similar process.

The use of a refraction table will be more convenient if, in constructing it, we replace $\sec^2 z$ by $1+\tan^2 z$ and, in the table giving $\log T$ and $\log b$ as functions of the temperature and pressure, multiply $\log T$ and $\log b$ by the constant factors

$$+\nu_0+\tfrac{1}{2}\alpha_0 \text{ and } +\tfrac{1}{2}\alpha_0$$

respectively. Then we may put, with Bessel,

$$\left.\begin{aligned}\lambda &= 1+(\nu_0+\tfrac{1}{2}\alpha_0)\tan^2 z\\ A &= 1+\tfrac{1}{2}\alpha_0\tan^2 z\end{aligned}\right\}, \qquad (53)$$

and tabulate λ and A as functions of z.

We now collect the logarithms of all the factors which enter into the complete expression for the refraction,

$$R = \mathrm{am}\tan z,$$

as follows:

1. The logarithm of the refraction under standard conditions or

$$\log \mathrm{a}_0\mathrm{m}_0 \tan z,$$

where

$$a_0 = 58''{\cdot}297g,$$

but is subject to correction from observations, and

$$m_0 = 1 - m_{1,0}\sec^2 z + m_{2,0}\sec^4 z - \ldots,$$

the values of the coefficients being taken from (39) with the standard values of ν and α.

2. The logarithm of the factor T, given in (44), and tabulated as a function of the observed temperature. This logarithm is to be multiplied by the factor

$$\lambda = 1 + 0{\cdot}001\,44\tan^2 z + \text{etc.}$$

3. Log b in (45) multiplied by the factor

$$A = 1 + 0{\cdot}000\,14\tan^2 z + \text{etc.}$$

4. Log t'', from (46), multiplied by the same factor.

It is to be remarked that the values of the factors λ and A are here not completely given, but only their first terms.

The preceding includes all that is necessary to the understanding and intelligent application of the formulae and tables of refraction. The completion of the fundamental theory with a view of perfecting the fundamental base of the tables requires an investigation of refraction near the horizon, the effect of humidity, and an extensive discussion of observations, none of which can be undertaken in the present work.

NOTES AND REFERENCES TO REFRACTION.

There is, perhaps, no branch of practical astronomy on which so much has been written as on this and which is still in so unsatisfactory a state. The difficulties connected with it are both theoretical and practical. The theoretical difficulties, with which alone we are concerned in the present work, arise from the uncertainty and variability of the law of diminution of the density of the atmosphere with height, and also from the mathematical difficulty of integrating the equations of the refraction for altitudes near the horizon, after the best law of diminution has been adopted. The list of modern writers on the subject includes many of the greatest names in theoretical and practical astronomy, extending from the time of Laplace to the present. Among those who have most contributed to the advance of the subject are,—Bouguer, Bradley, Laplace, Bessel, Young, Schmidt, Ivory, Gyldén and Radau.

BRUHNS, *Die Astronomische Strahlenbrechung*, Leipzig, 1861, gives an excellent synopsis of writings on the subject down to the time of its publication. Of these, the papers of Ivory, *On the Astronomical Refraction*, Philosophical Transactions for 1823 and 1838, are still especially worthy of study.

Since that time the following Memoirs are those on which tables of refraction have been or may be based :

GYLDÉN, *Untersuchungen über die Constitution der Atmosphäre und die Strahlenbrechung in derselben*, St. Petersburg, 1866–68.

There are two papers under this title published in the Memoirs of the St. Petersburg Academy : Série vii., Tome x., No. 1, and Tome xii., No. 4.

They contain the basis of the investigations on which the Poulkova tables of refraction were based. They are supplemented by :

Beobachtungen und Untersuchungen über die Astronomische Strahlenbrechung in der Nähe des Horizontes von V. FUSS ; St. Petersburg Memoirs, Série vii., Tome xviii., No 3.

RADAU'S Memoirs are :

Recherches sur la théorie des Réfractions Astronomiques ; Annales de l'Observatoire de Paris, Mémoires, Tome xvi., 1882.

Essai sur les Réfractions Astronomiques ; Ibid., Tome xix., 1889.

The latter work is devoted especially to the effect of aqueous vapour in the atmosphere, and contains tables for computing the refraction.

Among the earliest refraction tables which may still be regarded as of importance are those of Bessel in his *Fundamenta Astronomiae.* They were based upon the observations of Bradley. Bessel felt some doubt of the constant of refraction adopted in these tables, which was increased by uncertainty as to the correctness of Bradley's thermometer. The results of his subsequent researches are embodied in new tables found in the *Tabulae Regiomontanae*, where the constant of refraction of the *Fundamenta* was increased. These tables, enlarged and adapted to various barometric and thermometric scales, have formed the base of most of the tables used in practical astronomy to the present time. But, it has long been known that the constant of refraction adopted in them requires a material diminution—in fact, that the increase which Bessel made to the constant of the *Fundamenta* was an error.

In 1870 were published the Poulkova tables, based on the researches of Gyldén already quoted, under the title :

Tabulae Refractionum in usum Speculae Pulcovensis Congestae, Petropoli, 1870.

These tables give refractions less by ·002 85 of their whole amount than those of Bessel. Yet, the most recent discussions and comparisons indicate a still greater diminution to the constant.

In this connection it is to be remarked that up to the present time no account has been taken in using tables of refraction of the effect of the differences between the intensities of gravity in different latitudes. Even if the Poulkova tables are correct for the latitude of that point, 60°, their constant will still need a diminution at stations nearer the equator.

CHAPTER IX.

PRECESSION AND NUTATION.

Section I. Laws of the Precessional Motions.

119. The Equinox, or the point of intersection of the ecliptic and equator, may also be defined as a point 90° from the pole of each of these circles. Hence, if we mark on the celestial sphere

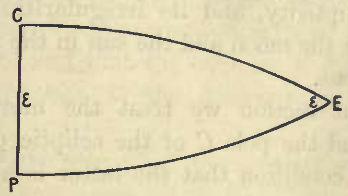

FIG. 21.

P, the north pole of rotation of the earth, or the celestial pole;
C, the pole of the ecliptic;
E, the equinox,

these points will be the vertices of a birectangular spherical triangle, of which the base *PC* is equal to the obliquity of the ecliptic.

Both the poles *P* and *C* are continuously in motion. Hence the equinox is also continuously in motion.

The motion of the ecliptic, or of the plane of the earth's orbit, is due to the action of the planets on the earth as a whole. It is very slow, at present less than half a second per year; and its direction and amount change but little from one century to the next.

The motion of the equator, or of the celestial pole, is due to the action of the sun and moon upon the equatorial protuberance of the earth. The theory of this action is too extensive a subject to be developed in the present work, belonging, as it does, to the domain of theoretical astronomy. We must, therefore, limit ourselves, at present, to a statement of the laws of the motion as they are learned from a combination of theory and observation. The motion is expressed as the sum of two components. One of these components consists in the continuous motion of a point, called the *mean pole* of the equator, round the pole of the ecliptic in a period of about 26 000 years, which period is not an absolutely fixed quantity. The other component consists in a motion called *nutation*, which carries the actual pole around the mean pole in a somewhat irregular curve, approximating to a circle with a radius of 9″, in a period equal to that of the revolution of the moon's node, or about 18·6 years. This curve has a slight ellipticity, and its irregularities are due to the varying action of the moon and the sun in the respective periods of their revolutions.

In the present section we treat the motion of the mean pole P. This, and the pole C of the ecliptic, determine a mean equinox, by the condition that the latter is always 90° distant from each.

Precession is the motion of the mean equinox, due to the combined motion of the two mean poles which determine it.

That part of the precession which is due to the motion of the pole of the earth is called *luni-solar*, because produced by the combined action of the sun and moon. It is commonly expressed as a sliding of the equinox along some position of the ecliptic considered as fixed.

That part which is due to the motion of the ecliptic is called *planetary*, because due to the action of the planets.

The combined effect of the two motions is called the *general precession*.

There is no formula by which the actual positions of the two poles can be expressed rigorously for any time. But their instantaneous motions, which appear as derivatives of the elements of position relative to the time, may be expressed numerically through a period of several centuries before or after any epoch. By the numerical integration of these expressions the actual positions may be found.

120. Fundamental conceptions.

In our study of this subject the two correlated concepts of a great circle and its pole, or of a plane and the axis perpendicular to it, come into play. In consequence of this polar relation, each quantity and motion which we consider has two geometrical representations in space, or on the celestial sphere. In treating the subject we shall begin in each case with that concept which is most easily formed or developed. This is commonly the pole of a great circle rather than the circle itself. In the case of the equator the primary concept is that of the celestial pole, since it is the axis of rotation of the earth which determines the equator. We note especially in this connection that CP is an arc of the solstitial colure, and that the equinox E is its pole. Either of these may be taken as the determining concept for the equinox.

The motion of the pole P at any instant may be conceived as taking place on a great circle G joining two consecutive positions of P. The polar plane and great circle of P then rotate around the axis and pole of G as a rotation axis, and the angular movement is the same as that of the pole P.

If G remains fixed as the plane moves, the rotation axis of the polar plane also remains fixed. But if the pole moves on a curve other than a great circle, the rotation axis moves also, rotating around the instantaneous position of the moving pole as a centre.

121. Motion of the celestial pole.

C and P being the respective poles of the ecliptic and equator, the law of motion of the pole of the equator, as derived from mechanical theory, is:

The mean pole moves continually toward the mean equinox *of the moment, and therefore at right angles to the colure CP, with speed n given by an expression of the form*

$$n = \text{P} \sin \epsilon \cos \epsilon, \qquad (1)$$

P *being a function of the mechanical ellipticity of the earth, and of the elements of the orbits of the sun and moon, and* ϵ *the obliquity of the ecliptic.*

P is subject to a minute change, arising from the diminution of the eccentricity of the earth's orbit; but the change is so slight that, for several centuries to come, it may be regarded as an absolute constant. The writer has called it the *precessional constant.** Taking the solar year as the unit of time, its adopted value is

$$\text{P} = 54''\!\cdot\!9066. \qquad (2)$$

Its rate of change is only $-0''\!\cdot\!000\,036\,4$ per century.

The centre C of the motion thus defined is the instantaneous position of the pole of the ecliptic at the moment. This pole is continually in motion in the direction CC', as shown by the dotted line in Figure 22. Hence, at the present time, the pole of

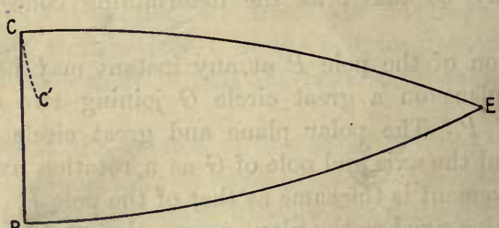

FIG. 22.

the ecliptic is approaching that of the equator. It follows from the law as defined that if the pole of the ecliptic were fixed in position, the obliquity would be constant. But, as the pole moves, it does not carry the pole of the earth with it, the motion of the latter being determined by the instantaneous position of the pole C, unaffected by its motion. Because the pole C is

* This term has been also applied to what may be called the mechanical ellipticity of the earth, or the ratio of excess of its polar over its equatorial moment of inertia to the polar moment.

at present diminishing its distance from P, the obliquity of the ecliptic is also diminishing.

The speed n of the motion of the pole P, as we have expressed it, is measured on a great circle. To find the angular rate of motion round C as a centre, we must divide it by $\sin \epsilon$, which will give the speed of luni-solar precession. We therefore have, still taking the year as the unit of time:

$$\left.\begin{array}{ll} \text{Annual motion of } P\text{, actual;} & n = 54''\cdot9066 \sin\epsilon \cos\epsilon \\ \text{Resulting luni-solar precession;} & p = 54''\cdot9066 \cos\epsilon \end{array}\right\} \ldots(3)$$

Neither n nor p is an absolute constant, since they both change with ϵ, the obliquity of the ecliptic.

122. Motion of the ecliptic.

Although the position of the ecliptic is to be referred to the equator and the equinox, so that the motion of the latter enters into the expression for that position, yet the actual motion of the ecliptic is independent of that of the equator. We, therefore, begin by developing the position and motion of the ecliptic, taking its position at some fixed epoch as a fundamental plane. Any such position of its plane is called the *fixed ecliptic* of the date at which it has that position.

The curve CC' along which the pole of the ecliptic is moving in our time is not a great circle, but a curve slightly convex toward the colure CP. To make clear the nature and effect of this motion we add Fig. 23, showing the correlated motion of the ecliptic itself. This represents a view of the ecliptic seen from the direction of its north polar axis. The positions of the poles P and C are reversed in appearance, because in Fig. 22 they are seen as from within the sphere, while in Fig. 23 they are seen as from without.

We shall now explain the motion by each of these correlated concepts. As the pole C moves, the ecliptic rotates around an axis NM (Fig. 23) in its own plane, determined by the condition that N is a pole of the great circle joining two consecutive positions of the pole C. From the direction of the motion it will be seen that the axis N, which we have taken as fundamental, is at each moment the descending node of the ecliptic, while M is the ascending node. The curve CC' being convex toward CP, the node N is slowly moving in the retrograde direction from E toward L.

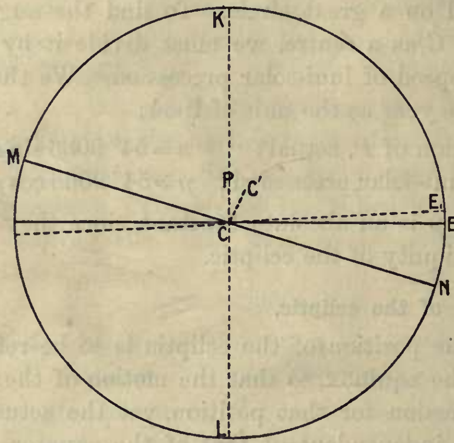

FIG. 23.

Fig. 24 shows the effect of this motion of the ecliptic upon the position of the equinox, supposing the equator to remain fixed. Here EN is the ecliptic and EQ the equator, as seen from the centre of the sphere, the observer at C in Fig. 23 looking in the direction E. The rotation of the ecliptic around N is con-

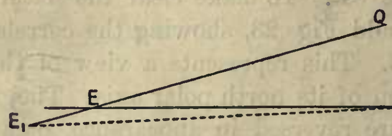

FIG. 24.

tinuous from NE toward NE_1, causing the equinox on the fixed equator to move in the positive direction EE_1, thus increasing the angle EN. This motion is that of planetary precession.

In consequence of luni-solar precession the colure CP is rotating around the instantaneous position of C as an axis, carrying with it its pole, the equinox E, in the direction EL with a motion yet more rapid than that of N. The angle ECN is therefore diminishing.

The instantaneous motion of the ecliptic is defined by the speed of its rotation around the axis MN, which speed we call κ, and by the position of N relative to the equinox. We put:

N_0, the angle between the direction of motion CC', as seen in Fig. 22, and some fixed position of the colure, say that of 1850, which we call the initial colure and date. The correlated concept is the arc E_1N (Figs. 23 and 24).

N, the angle between this direction at the epoch t and the colure at t. This is equivalent to saying that N_0 is the angle which the tangent to the curve CC' makes with the colure of the initial date, while N is the angle which it makes with the actual moving colure. The correlated N is the arc EN (Figs. 23 and 24).

These quantities determine only the instantaneous motion, not the actual position of the ecliptic. To express the latter we shall hereafter put

k, the angle CC' (Fig. 22) $= ENE_1$ (Fig. 24), which the actual ecliptic at any epoch makes with the initial ecliptic or fundamental plane.

N_1, the angle which the node of the actual ecliptic makes with the initial line of the equinoxes.

In the usual method of expressing the position of the moving with respect to a fixed ecliptic, k is the inclination, and $180° - N_1$ the longitude of the ascending node, referred to the initial equinox. The value of N at present being 6° and a fraction, the longitude of this node is 173° and a fraction.

123. Numerical computation of the motion of the ecliptic.

Proceeding to the numerical computation, the speed of the instantaneous motion and the values of N_0 are found by theory to be as follows at three epochs, of which the extremes are 250 years before and after 1850.*

Epoch.	log κ.	κ.	N_0.	$\kappa \cos N_0$.	$\kappa \sin N_0$.	
1600	1·675 00	47″·316	5° 17′·96	47·113	4·370	
1850	1·673 40	47 ·141	6 30 ·32	46·838	5·341	(4)
2100	1·671 87	46 ·976	7 42 ·82	46·550	6·305	

* *Astronomical Papers of the American Ephemeris*, vol. iv.; *Elements and Constants*, p. 186.

Our next step is to derive from (4) the actual position of the ecliptic, at any intermediate epoch. This we do by referring the position of the pole C to rectangular coordinates, the curvature of the sphere within so minute a region as that over which the motion extends being insensible. Taking CP as the axis of Y and x, y as the coordinates of C, we shall have

$$\left.\begin{array}{l}\kappa \sin N_0 = D_t x \\ \kappa \cos N_0 = D_t y\end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Putting T for the time in centuries after 1850, the three values of these quantities already given may be developed in the form:

$$D_T x = 5''{\cdot}341 + 0''{\cdot}3870T - 0''{\cdot}000\,56T^2,$$
$$D_T y = 46''{\cdot}838 - 0''{\cdot}1126T - 0''{\cdot}001\,04T^2.$$

Then, by integration,

$$\left.\begin{array}{l}x = 5''{\cdot}341T + 0''{\cdot}1935T^2 - 0''{\cdot}000\,19T^3 \\ y = 46''{\cdot}838T - 0''{\cdot}0563T^2 - 0''{\cdot}000\,35T^3\end{array}\right\} \quad \ldots\ldots\ldots(6)$$

Here x and y are the coordinates of the pole C referred to the colure of 1850 as a fixed direction. To find the polar coordinates, we put

C', the position of the pole at any epoch;
k, the arc of the great circle CC';
N_1, the angle PCC'.

The values of k and N_1 at any time are then found from the equations

$$k \sin N_1 = x,$$
$$k \cos N_1 = y.$$

Computing the values of x and y from (6) for epochs fifty years apart, we have the results shown in the following table:

Epoch.	x.	y.	k.	N_1.	
1750	− 5″·147	− 46″·894	− 47″·176	6° 15′·81	
1800	− 2 ·622	− 23 ·433	− 23 ·579	6 23 ·07	
1850	0 ·000	0 ·000	0 ·000	6 30 ·32	
1900	+ 2 ·719	+ 23 ·405	23 ·562	6 37 ·55	(7)
1950	5 ·534	46 ·781	47 ·107	6 44 ·79	
2000	8 ·446	70 ·129	70 ·636	6 52 ·04	
2050	11 ·454	93 ·448	94 ·147	6 59 ·28	
2100	14 ·558	116 ·738	117 ·642	7 6 ·52	

In this table the value of N_1 for the initial epoch 1850 is the direction of the instantaneous motion at that epoch. For convenience in subsequent computation the value of k is regarded as negative before 1850, thus avoiding a change of 180° in N_1.

124. Combination of the precessional motions.

We have now to combine the two motions which we have defined, so as to obtain the general precession. We begin, as before, with the speeds of the motions and not with their total amount between two epochs. This speed is given by the motion during a time so short that we may regard the motion as infinitesimal, but may be expressed with reference to any unit of time that we find convenient.

If we define the motion by that of the two poles, the annual general precession is equal to the annual change in the direction of the colure PC, as measured by the rotation around the point C. But the effect of the combined motions on the position of the actual equinox can best be studied by transferring our field of view from the region of the poles to that of the equinox, and studying the motion of the ecliptic and equator themselves instead of the motion of their poles.

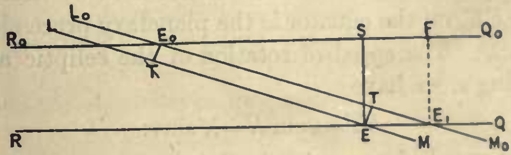

FIG. 25.

Fig. 25 is a view of the moving equinox, seen from the same view-point as in Fig. 24, but infinitely magnified.

In Fig 25, let us have:

QR, the position of the equator;
LM, that of the ecliptic;
E, the equinox.

Two positions of each of these are marked, the one set Q_0R_0; L_0M_0; E_0; for the origin or zero of time; the other, QR; LM; E;

after a period of time which we regard as infinitesimal. All the segments in the figure are, therefore, treated as infinitesimals, and are considered to represent speeds of motion, each speed being multiplied by dt.

We now apply what has already been said of the motion of the poles to the figure, with the following results:

The two equators Q_0R_0 and QR intersect at points 90° in either direction from the region shown in the figure, and their infinitesimal arcs shown in the figure are parallel.

The perpendicular distance ES of the two equators from each other is equal to ndt; but, in accordance with what has just been said, we may consider this distance to represent n itself, the factor dt being dropped.

The two ecliptics L_0M_0 and LM intersect at the point N, which cannot be marked in the figure, lying in the direction LM at a distance from E_0 (or E) represented by the angle N already defined.

The speed p of the luni-solar precession is represented by the arc E_0E_1 between the intersections of the two equators with the fixed ecliptic L_0M_0.

The arc FE_0 may be called the luni-solar precession in R.A. Its value is $p \cos \epsilon$ or $\text{P} \cos^2\epsilon$, but it is not used by itself.

The arc EE_1 on the equator is the planetary precession in R.A. We call it λ'. The speed of rotation of the ecliptic around the node N being κ, we have

$$ET = \kappa \sin N = \lambda' \sin \epsilon.$$

The total speed of precession in R.A. is

$$E_0S = E_0F - EE_1 = \text{P} \cos^2\epsilon - \lambda'.$$

The general precession is defined as the motion of the equinox E along the moving ecliptic. It is measured by its projection E_0T, which differs from E_0E only by an infinitesimal of the second order. Its two parts are $p = E_0E_1$ and $E_1T = \lambda' \cos \epsilon$ taken negatively. We call its speed l. Hence

$$l = p - \lambda' \cos \epsilon = (\text{P} - \lambda') \cos \epsilon. \quad \text{.................}(8)$$

From the law of motion of the equator, P always moving at right angles to CP, it will be seen that the instantaneous change of the obliquity is due wholly to the motion of the ecliptic, and may be found by resolving the instantaneous motion of C into two rectangular components, one in the direction CP; the other in the direction CE. (Figs. 22, 23.)

Since, by the preceding notation,

$$N = \text{angle } PCC',$$

$$\kappa = \text{rate of motion of } C,$$

we shall have $$D_t\epsilon = -\kappa \cos N.$$

125. Expressions for the instantaneous rates of motion.

As the conceptions developed in the preceding sections are fundamental in spherical astronomy, we recapitulate them. Dropping the factor dt and supposing the lines in the figure to represent rates of motion, the perpendicular distance SE or FE_1 at the equinox between the two positions of the equator will represent n. The distance ET between the ecliptics will represent $\kappa \sin N$. We then have

speed of luni-solar precession in longitude,

$$p = E_0E_1 = \text{P} \cos \epsilon; \quad \text{.........................(9)}$$

speed of planetary precession in longitude,

$$-\lambda' \cos \epsilon = E_1T = -\kappa \sin N \cot \epsilon; \quad \text{.............(10)}$$

speed of general precession in longitude,

$$l = p - \lambda' \cos \epsilon = (P - \lambda') \cos \epsilon; \quad \text{.................(11)}$$

speed of luni-solar precession in R.A.,

$$E_0F = N_1E = p \cos \epsilon = \text{P} \cos^2\epsilon; \quad \text{.................(12)}$$

speed of planetary precession in R.A.,

$$-\lambda' = EE_1 = -\kappa \sin N \operatorname{cosec} \epsilon; \quad \text{.................(13)}$$

speed of general precession in R.A.,

$$m = \text{P} \cos^2\epsilon - \lambda'; \quad \text{.........................(14)}$$

speed of change of the obliquity of the ecliptic,

$$D_t\epsilon = -\kappa \cos N. \quad \text{.........................(15)}$$

126. Numerical values of the precessional motions and of the obliquity.

We shall now compute from the data already given, and the preceding formulae, the actual speeds of the various precessional motions for some fundamental epochs. We have all the data for 1850 at hand; but, for the other epochs, it is necessary to use the results for 1850 to compute the data. These are the values of P and N already given, and the obliquity, of which the value for 1850 is

$$\epsilon = 23° \, 27' \, 31''\cdot 68.$$

For this class of computations the century is the most convenient unit of time; we therefore multiply the value of P by 100, so that

$$\text{P} = 5490''\cdot 66.$$

The computation is as follows:

Epoch 1850.

log	P	3·739 624 5
,,	$\cos\epsilon$	9·962 533 4
log	p	3·702 157 9
,,	$\sin\epsilon$	9·599 980 8
,,	$p\cos\epsilon$	3·664 691 3
,,	n	3·302 138 7 (where n = p sin ε)

Luni-solar precession $p = 5036''\cdot 84$

Luni-solar precession in R.A. $= 4620\ \cdot 53$

$n = 2005\ \cdot 11$

$= 133^s\cdot 674$

log	$\cos N_0$	9·997 19
,,	κ	1·673 40
,,	$\sin N_0$	9·054 21
,,	$\operatorname{cosec}\epsilon$	0·400 02
log	λ'	1·127 63
,,	$\lambda'\cos\epsilon$	1·090 16

$\lambda' = 13''\cdot 416$

$\lambda'\cos\epsilon = 12\ \cdot 307$

General precession in long. $l = 5024''\cdot 53 = 83'\cdot 742$

General precession in R.A. $m = 4607\ \cdot 11 = 307^s\cdot 141$

$\log\kappa\cos N_0 = 1\cdot 670\ 59$

$D_t\epsilon = -46''\cdot 837$

We have next to derive the data and compute the speeds of motion for the extreme fundamental epochs. N_0 being the angular distance of the instantaneous axis of rotation from the equinox of 1850, and N that from the actual equinox, it follows that their speeds differ by the general precession in longitude, so that we have

$$D_tN = D_tN_0 - l.$$

By developing the values (4) of N_0, we have

$$N_0 = 6°\ 30'\cdot 32 + 28'\cdot 972T + 0'\cdot 011T^2.$$

We have just found $\quad l = 83'\cdot 742.$

Therefore, postponing terms in T^2, we have

$$N = 6°\ 30'\cdot 32 - 54'\cdot 770T,$$

from which we derive N for other epochs.

With these expressions, and the values of κ derived from (4) by interpolation, we compute the following values of the quantities required to find the obliquity of the ecliptic and the planetary precession:

Epoch.	N.	log κ.	κ cos N.	κ sin N.	
1750	7° 25′·09	1·674 03	46″·814	6″·094	
1800	6 57 ·71	1·673 71	46 ·828	5 ·718	
1850	6 30 ·32	1·673 40	46 ·837	5 ·341	
1900	6 2 ·94	1·673 09	46 ·844	4 ·964	......(16)
1950	5 35 ·55	1·672 78	46 ·849	4 ·587	
2000	5 8 ·17	1·672 47	46 ·851	4 ·211	
2050	4 40 ·78	1·672 17	46 ·851	3 ·835	
2100	4 13 ·40	1·671 87	46 ·848	3 ·459	

Differencing the preceding κ cos N, the centennial variation of the obliquity, we find that its second differences are appreciably constant, and that its values may be developed in the form,

$$\kappa \cos N = -D_T\epsilon = 46''\cdot 837 + 0''\cdot 017T - 0''\cdot 0051T^2.$$

By integrating and adding as a constant the obliquity for 1850, we find

$$\epsilon = 23°\ 27'\ 31''\cdot 68 - 46''\cdot 837T - 0''\cdot 0085T^2 + 0''\cdot 0017T^3.$$

This gives the following values of the obliquity for the eight epochs from 1750 to 2100:

Epoch.	ϵ.	log sin ϵ.	log cos ϵ.	
1750	23° 28′ 18″·51	9·600 207 9	9·962 490 7	
1800	23 27 55 ·10	9·600 094 3	9·962 512 1	
1850	23 27 31 ·68	9·599 980 8	9·962 533 4	
1900	23 27 8 ·26	9·599 867 1	9·962 554 8	......(17)
1950	23 26 44 ·84	9·599 753 4	9·962 576 2	
2000	23 26 21 ·41	9·599 639 6	9·962 597 6	
2050	23 25 57 ·99	9·599 525 9	9·962 619 0	
2100	23 25 34 ·56	9·599 412 1	9·962 640 5	

These quantities give the data necessary for the computation of all the precessional motions for the several epochs. Another approximation to the values for epochs before and after 1850 may be made by using the varying values of l to derive fresh values of N. But this revision will not appreciably change the results which, so far as necessary for use during the present century, will be found in Appendix III.

Section II. Relative Positions of the Equator and Equinox at Widely Separated Epochs.

127. Our next problem is, from the instantaneous motions just found, to define the actual position of the equator at any one epoch T relative to its position at some other epoch T_0. We regard T_0 as a constant, and term it the initial epoch. The other, with the quantities which depend upon it, are treated as variable.

Let us consider the spherical quadrangle PP_0C_0C formed by the positions of the two poles at the two epochs.

In order to represent all the quantities on the figure, it has been necessary to draw it so as to express the motion over a period of several thousand years. In consequence of this the angle CC_0P, as represented in the figure, is negative, owing to the motion of the pole P having carried the arc C_0P over the point C. The student who wishes to do so can easily draw the figure and apply the numbers for the period through which the computations actually extend.

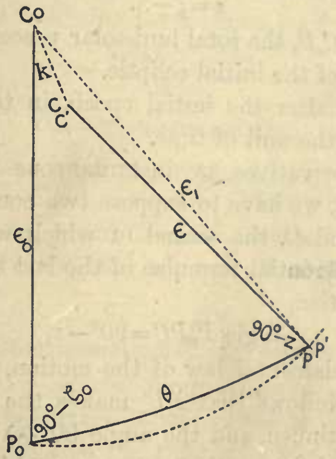

FIG 26.

We divide the quadrangle into two triangles by the diagonal C_0P, and then have or put

$\epsilon_0 = C_0P_0$, the obliquity of the ecliptic at the initial epoch.

$\epsilon_1 = C_0P$, the obliquity of the equator of the epoch T to the initial ecliptic.

θ, the arc P_0P joining the two positions of the pole. This arc is to be taken as that of a great circle, not the actual path of P, which is represented by the dotted arc.

k, the arc C_0C, through which the pole of the ecliptic has moved.

N_1, the angle P_0C_0C.

ζ, the amount by which the angle C_0PP_0 falls short of 90°.

ζ_0, the amount by which the angle C_0P_0P falls short of 90°.

λ, the angle C_0PC, which is equal to the total planetary precession on the equator of the epoch T, or to the arc of this equator intercepted between the two ecliptics, taken negatively in the figure.*

*It should be noted that the angle λ, when taken positively, as is done in the present work for dates subsequent to the initial epoch, is subtractive from the lunar solar precession during the next 500 years. Its value will reach a maximum

z, the amount by which the angle P_0PC falls short of 90°, so that we have

$$z = \zeta - \lambda.$$

ψ, the angle P_0C_0P, the total luni-solar precession on the fixed position of the initial ecliptic.

T, the interval after the initial epoch, in terms of 100 solar years as the unit of time.

To find the derivatives, or instantaneous motions of these various quantities, we have to suppose two consecutive positions of the poles P and C, the second of which we call P' and C', and apply the differential formulae of the last section.

Since, by definition,

$$\text{Angle } P_0PC = 90° - z,$$

while, by the fundamental law of the motion, P moves at right angles to CP, it follows that PP' makes the angle z with the direction P_0P continued, and the angle $90° + \lambda$ with the arc PC'. Its rate of motion being n, the space over which it moves in an infinitesimal time is ndt. Hence

$$PP' = ndt = \text{P} \sin \epsilon \cos \epsilon dt \quad (\S\,121).$$

We then have, as the effect of the motion on θ and ϵ_1,

$$\left.\begin{aligned} d\theta &= P_0P' - P_0P = n \cos z dt \\ d\epsilon_1 &= C_0P' - C_0P = n \sin \lambda dt \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (18)$$

The instantaneous motion of ψ is the angle subtended by PP', or the motion ndt, as seen from C_0. We therefore have, by the theorems of differential spherical trigonometry (§ 7),

$$ndt \cos \lambda = \sin C_0P\, d\psi = \sin \epsilon_1 d\psi.$$

Hence
$$D_t\psi = \frac{n \cos \lambda}{\sin \epsilon_1}.$$

Between the parts of the triangle C_0PC we have the relation

$$\frac{\sin P}{\sin C_0C} = \frac{\sin C_0}{\sin CP},$$

or
$$\frac{\sin \lambda}{\sin k} = \frac{\sin(N_1 - \psi)}{\sin \epsilon}.$$

---

about 2150 and then diminish until about 2400, when it will become negative through the arc C_0P crossing C_0C. It should also be seen that the precessional motion λ' defined in the preceding section, is the speed of λ at the initial epoch, but is not rigorously equal to that speed at other epochs.

Owing to the minuteness of λ and k, we may take their sines and arcs as equal, and put $\cos\lambda = 1$. Substituting for n its value, we have the following three equations for λ, ϵ_1, and ψ:

$$\left.\begin{aligned} \lambda &= \frac{k \sin (N_1 - \psi)}{\sin \epsilon} \\ D_t\epsilon_1 &= \text{P} \sin \epsilon \cos \epsilon \sin \lambda \\ D_t\psi &= \frac{\text{P} \sin \epsilon \cos \epsilon}{\sin \epsilon_1} = \frac{\text{P} \sin 2\epsilon}{2 \sin \epsilon_1} \end{aligned}\right\} \qquad (19)$$

The only unknown quantities in these equations are λ, ϵ_1, and ψ, the values of which we are now to derive by successive approximations.

128. Numerical approximations to the position of the pole.

At first we need only a rough value of λ to be used in finding ϵ_1. For this purpose it will suffice to suppose ψ to vary uniformly with its motion for 1850, which is $\text{P} \cos \epsilon_0$. We may therefore put, in the first equation,

$$\psi = 5034''{\cdot}7\,T = 83'{\cdot}9\,T.$$

With the values of ψ computed from this expression, and the values found in §§ 123 and 126 for k, N_1, and ϵ, we compute approximate values of λ as follows:

1850	$\psi = 0°\ 0'{\cdot}0$	$\lambda = 0''{\cdot}00$
1900	0 42 ·0	6 ·11
1950	1 23 ·9	11 ·03
2000	2 5 ·9	14 ·76
2050	2 47 ·8	17 ·30
2100	3 29 ·8	18 ·65

Changing the values of λ to arc, and taking the 6th decimal as a unit, we find that $\sin\lambda$ may be expressed in the form:

$$10^6 \sin \lambda = 65{\cdot}03T - 11{\cdot}54T^2.$$

Substituting this expression in (19), we have

$$D_t\epsilon_1 = 0''{\cdot}1304T - 0''{\cdot}0232T^2,$$

and by integration,

$$\epsilon_1 = \epsilon_0 + 0''{\cdot}0652T^2 - 0''{\cdot}0077T^3. \qquad (20)$$

With the values of ϵ_1 derived from this expression, which differ by only a small fraction of a second from ϵ_0, we find from the third equation (19), using the values of ϵ in (17), the following values of the differential variation of ψ with its differences:

Epoch.	$\frac{d\psi}{dT}$.	Δ_1.	Δ_2.
1750	5038″·97		
		– 1″·06	
1800	5037 ·91		– ·01
		– 1 ·07	
1850	5036 ·84		+ ·01
		– 1 ·06	
1900	5035 ·78		– ·01
		– 1 ·07	
1950	5034 ·71		– ·01
		– 1 ·08	
2000	5033 ·63		·00
		– 1 ·08	
2050	5032 ·55		– ·01
		– 1 ·09	
2100	5031 ·46		

.........(21)

These values may be developed in the form

$$\frac{d\psi}{dt} = 5036''{\cdot}84 - 2''{\cdot}130T - 0''{\cdot}010T^2.$$

Hence, by integration,

$$\left.\begin{aligned}\psi &= 5036''{\cdot}84T - 1''{\cdot}065T^2 - 0''{\cdot}003T^3\\ &= 83'{\cdot}947T - 0'{\cdot}0177T^2\end{aligned}\right\} \quad \text{.........(22)}$$

This is the definitive expression for the total precession of the equinox upon the fixed ecliptic of 1850.

129. Numerical value of the planetary precession.

We now have all the data for computing the definitive values of λ from the first equation (19), as follows:

Epoch.	ψ.	$N_1 - \psi$.	λ.	Δ_1.	Δ_2.
1750	– 1° 23′·96	7° 39′·77	– 15″·794		
				+ 8″·491	
1800	– 0 41 ·98	7 5 ·04	– 7 ·303		– 1″·188
				7 ·303	
1850	0 ·00	6 30 ·32	0 ·000		– 1 ·190
				6 ·113	
1900	+ 0 41 ·97	5 55 ·58	+ 6 ·113		– 1 ·192
				4 ·921	
1950	1 23 ·93	5 20 ·86	11 ·034		– 1 ·191
				3 ·730	
2000	2 5 ·88	4 46 ·16	14 ·764		– 1 ·193
				2 ·537	
2050	2 47 ·82	4 11 ·46	17 ·301		– 1 ·194
				1 ·343	
2100	3 29 ·76	3 36 ·75	18 ·644		

The values of λ may be developed in the form:

$$\lambda = 13''{\cdot}416T - 2''{\cdot}380T^2 - 0''{\cdot}0014T^3. \qquad (23)$$

This expression represents all the above special values within $0''{\cdot}001$.

130. Auxiliary angles.

For the angles ζ_0 and ζ, we have, from Napier's analogies, applied to the triangle P_0C_0P:

$$\tan\tfrac{1}{2}(\zeta+\zeta_0) = \frac{\cos\frac{1}{2}(\epsilon_1+\epsilon_0)\tan\frac{1}{2}\psi}{\cos\frac{1}{2}(\epsilon_1-\epsilon_0)},$$

$$\tan\tfrac{1}{2}(\zeta-\zeta_0) = \frac{\sin\frac{1}{2}(\epsilon_1-\epsilon_0)\cot\frac{1}{2}\psi}{\sin\frac{1}{2}(\epsilon_1+\epsilon_0)}.$$

The arc $\epsilon_1 - \epsilon_0$ is only a small fraction of $1''$. We may therefore neglect its powers. Putting

$$\Delta\epsilon_1 = \epsilon_1 - \epsilon_0,$$

the above equations may be written

$$\left.\begin{aligned}\tan\tfrac{1}{2}(\zeta+\zeta_0) &= \cos(\epsilon_0+\tfrac{1}{2}\Delta\epsilon_1)\tan\tfrac{1}{2}\psi\\ \tan\tfrac{1}{2}(\zeta-\zeta_0) &= \frac{\Delta\epsilon_1}{2\sin(\epsilon_0+\frac{1}{2}\Delta\epsilon_1)\tan\frac{1}{2}\psi}\end{aligned}\right\}. \qquad (24)$$

The computation of the first of these expressions involves no difficulty, all the quantities which enter into it having been found. The results are as follows:

Epoch.	$\frac{1}{2}\psi$.	$\frac{1}{2}(\zeta+\zeta_0)$.	Δ_1.	Δ_2.	Δ_3.
1750	$-2518''{\cdot}951$	$-2310''{\cdot}769$	$1155''{\cdot}512$		
				-255	
1800	$-1259\ {\cdot}343$	$-1155\ {\cdot}257$	$1155\ {\cdot}257$		$+9$
				-246	
1850	$0\ {\cdot}000$	$0\ {\cdot}000$	$1155\ {\cdot}011$		17
				-229	
1900	$+1259\ {\cdot}076$	$1155\ {\cdot}011$	$1154\ {\cdot}782$		6
				-223	
1950	$2517\ {\cdot}886$	$2309\ {\cdot}793$	$1154\ {\cdot}559$		17
				-206	
2000	$3776\ {\cdot}426$	$3464\ {\cdot}352$	$1154\ {\cdot}353$		18
				-188	
2050	$5034\ {\cdot}698$	$4618\ {\cdot}705$	$1154\ {\cdot}165$		
2100	$6292\ {\cdot}698$	$5772\ {\cdot}870$			

Developing these quantities in powers of T, we find

$$\zeta+\zeta_0 = 4620''{\cdot}53T - 0''{\cdot}984T^2 + 0''{\cdot}036T^3. \qquad (25)$$

The second of the equations (24) is subject to the inconvenience of giving $\zeta-\zeta_0$ as the quotient of two small quantities. This makes it necessary to express the numerator of the fraction in a form to facilitate its development. By eliminating $\sin\lambda$ from the first two equations (19), we find

$$D_t\epsilon_1 = \text{P}k \cos\epsilon \sin(N_1 - \psi). \quad\text{(26)}$$

In using this formula we express P in arc, and thus find the following special values of $D_t\epsilon_1$, with their differences:

Epoch.	$D_t\epsilon_1$.	Δ_1.	Δ_2.
1750	−0″·153 60		
		+8259	
1800	−0 ·071 01		−1158
		7101	
1850	0 ·000 00		1160
		5941	
1900	+0 ·059 41		1160
		4781	
1950	0 ·107 22		1159
		3622	
2000	0 ·143 44		1161
		2461	
2050	0 ·168 05		1161
		1300	
2100	0 ·181 05		

These values may be represented in the form:

$$D_t\epsilon_1 = 0''{\cdot}130\,42T - 0''{\cdot}023\,20T^2.$$

Then, by integration,

$$\Delta\epsilon_1 = 0''{\cdot}065\,21T^2 - 0''{\cdot}007\,73T^3, \quad\text{(27)}$$

which is the numerator of the fraction.

To express the denominator we may choose either of two methods. The simplest in form, but not the shortest numerically, is to compute the numerical values of the nearly constant quantity $\dfrac{T}{\tan\frac{1}{2}\psi}$ from the preceding values of $\frac{1}{2}\psi$ for 1750, 1950, and 2100, and develop them in the form $a + bT + cT^2$. The product of this development into the above value of $\Delta\epsilon$, multiplied by $\text{cosec}(\epsilon_0 + \Delta\epsilon_1)$, (from which we may drop $\Delta\epsilon_1$) will give $\zeta - \zeta_0$.

The other method is to develop the denominator and its reciprocal thus:

$$\tan\tfrac{1}{2}\psi = \tfrac{1}{2}\psi(1 + \tfrac{1}{12}\psi^2),$$

$$\frac{1}{\tan\frac{1}{2}\psi} = \frac{2}{\psi}(1 - \tfrac{1}{12}\psi^2).$$

We shall, in form, change the product

$$\Delta\epsilon_1 \cot \tfrac{1}{2}\psi \text{ into } T \cot \tfrac{1}{2}\psi \times \frac{\Delta\epsilon_1}{T}.$$

Changing the value (22) of ψ to arc, and, for convenience, expressing the result in units of the sixth place, we have

$$10^6\psi = 244\,19T - 5{\cdot}1T^2 - 0{\cdot}015T^3,$$

$$\tfrac{1}{12}\psi^2 = 0{\cdot}000\,05T^2,$$

$$\frac{1}{2\tan\frac{1}{2}\psi} = \frac{40{\cdot}952 + 0{\cdot}0086T - 0{\cdot}0020T^2}{T}.$$

The product of this into $2\Delta\epsilon_1$ gives

$$\frac{\Delta\epsilon_1}{\tan\frac{1}{2}\psi} = 5''{\cdot}3409T - 0''{\cdot}6320T^2 - 0''{\cdot}0004T^3.$$

Owing to the minuteness of the quantities in question, we may take $\zeta - \zeta_0$ as equal to its tangent, and neglect $\Delta\epsilon_1$ in the denominator of (24). We thus obtain the value of $\zeta - \zeta_0$ by multiplying the last equation by

$$\operatorname{cosec}\epsilon_0 = 2{\cdot}512; \quad \log = 0{\cdot}400\,02.$$

This gives

$$\zeta - \zeta_0 = 13''{\cdot}416T - 1''{\cdot}588T^2 - 0''{\cdot}0010T^3. \quad \ldots\ldots\ldots(28)$$

The combination of this expression with (23) and (25) gives the values of ζ, ζ_0, and z.

$$\left.\begin{aligned} \zeta &= 2316''{\cdot}97T - 1''{\cdot}286T^2 + 0''{\cdot}017T^3 \\ \zeta_0 &= 2303\ {\cdot}56T + 0\ {\cdot}302T^2 + 0\ {\cdot}018T^3 \\ z = \zeta - \lambda &= 2303\ {\cdot}55T + 1\ {\cdot}094T^2 + 0\ {\cdot}018T^3 \end{aligned}\right\}. \quad \ldots\ldots(29)$$

131. Computation of angle between the equators.

It only remains to develop θ. There are two ways of doing this, the agreement from the results of which will serve as a control of the correctness of the computation. The most ready method is to solve the triangle C_0P_0P, which gives

$$\sin\theta = \frac{\sin\epsilon_0 \sin\psi}{\cos\zeta}.$$

By this formula, using the values of ψ already found, we compute the following values of θ:

Epoch.	θ.	Δ_1.	Δ_2.	Δ_3.
1750	$-2005''\cdot50$			
		$+1002''\cdot84$		
1800	$-1002\cdot66$		-18	
		$1002\cdot66$		-4
1850	$0\cdot00$		-22	
		$1002\cdot44$		-2
1900	$+1002\cdot44$		-24	
		$1002\cdot20$		-3
1950	$2004\cdot64$		-27	
		$1001\cdot93$		-4
2000	$3006\cdot57$		-31	
		$1001\cdot62$		-3
2050	$4008\cdot19$		-34	
		$1001\cdot28$		
2100	$5009\cdot47$			

These values give the development

$$\theta = 2005''\cdot11T - 0''\cdot43T^2 - 0''\cdot041T^3. \qquad \text{(30)}$$

The other method of developing θ is from the equation (18), which gives

$$\frac{d\theta}{dT} = n \cos z = \tfrac{1}{2} P \sin 2\epsilon \cos z, \qquad \text{(31)}$$

which will lead to a result in agreement with the above.

By interpolation of the various quantities we have tabulated, the position of the equator and equinox at any epoch between 1750 and 2100 is found relatively to the positions for 1850 as the initial epoch. A similar computation may be made, using 1900, or other epochs, as the initial one. The results of such computations are tabulated in Appendix IV.

Section III. Nutation.

132. Motion of nutation.

The pole P, whose motion has just been developed, is the mean, not the actual pole. The latter moves round the mean pole with the motion called nutation, which arises in the following way:

When the theoretical expressions for the motion of the pole under the combined action of the sun and moon are derived from the theory of rotating bodies, it is found that this motion may be resolved into two, one depending on the longitudes of the sun and moon, and therefore periodic; the other independent of these longitudes, and therefore progressive. Each term expressive of a periodic motion, if taken separately, would bring the pole back to its original position at the end of a revolution of the sun or moon. In fact the largest component of this motion goes through two periods in one such revolution.

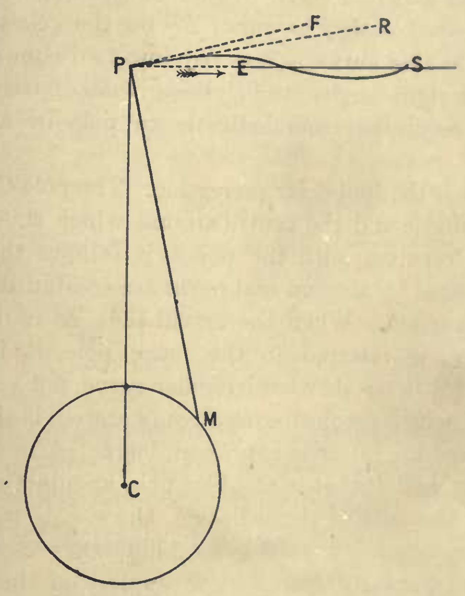

FIG. 27.

The progressive motion includes that of precession, described in the two preceding sections. But a part of this motion is dependent upon the longitude of the moon's node, which makes a revolution in 18·6 years. This part of the motion being periodic, is included in the nutation, and in fact forms much the largest term of the nutation.

To show the relation of these terms to precession, let P and C be the respective poles of the equator and ecliptic, and M that of the moon's orbit. Then the progressive motions produced by the two bodies are in the respective directions *PE* and *PF*, at right angles to *PC* and *PM* respectively. The lunar component *PF* is more than double the other. The resultant of the two is an instantaneous motion *PR*, making an angle with *PE*, which depends on the position of the pole *M* relative to *P* and *C*.

Now *M* revolves round *C* at a nearly invariable distance of 5°11′, in a period of 18·6 years. The result is that the direction *PR* of the motion of *P* continually oscillates on one side and the other of *PE* in a period equal to that of the moon's node, and *P* itself described a sinuous curve *PS* on the celestial sphere. The motion on this curve is resolved into two components, the one always at right angles to *PC*, being that of the mean pole, the other a revolution round the mean pole in a period of 18·6 years.

The former is the luni-solar precession. The pole *C* being both that of the ecliptic and the centre around which the pole of the moon's orbit revolves with the nodes, it follows that the precessions produced by the sun and moon are combined to produce the total precession. When the actual pole, as it describes its sinuous curve, is referred to the mean pole, it is found to revolve round it in a somewhat irregular curve, not very different from a circle, which revolution, as already stated, is the nutation.

To show how the latter is expressed, let P_0 be the position of the mean pole, and *P* that of the true pole at any time, while *C*, as before, is the pole of the ecliptic. Draw P_0Q perpendicular to CP_0. Then the angle at *C* is so minute, never reaching 20″, that we may regard P_0QP as a right angle, and the position of *P* relative to P_0 may then be expressed by the rectangular coordinates QP_0 and *QP*. But, for all the purposes of astronomy, it is convenient to use, instead of P_0Q, the angle P_0CQ, connected with it by the relation

$$P_0Q = P_0CP \sin \epsilon.$$

The maximum value of the distance P_0P between the poles is about 10″, a quantity so small in comparison with ϵ that their ratio may be treated as an infinitesimal, and the triangle P_0QP regarded as plane. Under this restriction the component *QP* of the nutation will be the change in the obliquity, and P_0CQ the change in the luni-solar precession, produced by the nutation. They are represented by $\Delta\epsilon$ and $\Delta\psi$, thus according with the notation of the last chapter.

The term of nutation depending on the moon's node is more than 12 times as large as the largest term depending on the sun's longitude, and more than 50 times as large as the largest depending on the moon's longitude. Its effects on obliquity and precession at the epoch 1900 are:

$$\left.\begin{aligned}\Delta\epsilon &= 9''{\cdot}21 \cos \Omega \\ \Delta\psi &= 17''{\cdot}234 \sin \Omega\end{aligned}\right\}, \quad \text{(32)}$$

Ω being the longitude of the moon's node.

Owing to the secular diminution of the obliquity these terms are effected by a minute secular variation, which need not be considered at present.

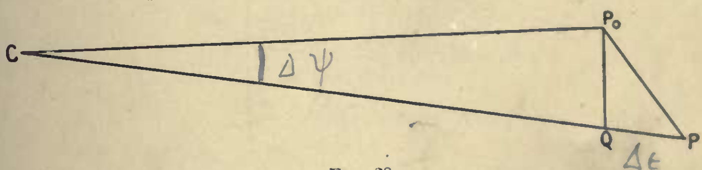

FIG. 28.

The coefficient $9''{\cdot}21$ of $\cos \Omega$ in the expression for ϵ is called the *constant of nutation*. The value here assigned is the mean result of all available observations up to 1895.*

133. Theoretical relations of precession and nutation.

In presenting the results of the theory of the relation between the motions of precession and nutation, the figure of the earth and the mass of the moon, I adopt the constant numerical coefficients from Oppolzer's exhaustive investigation.† We put

C, the moment of inertia of the earth around the axis of rotation.

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* *Elements and Constants*, pp. 130, 189, and 195. The results of a subsequent adjustment of the mass of the moon led to the theoretical value $9''{\cdot}214$, which, however, was not accepted by the Paris Conference of 1896.

† *Lehrbuch zur Bahnbestimmung der Planeten und Cometen*, 2nd edition, Leipzig, 1882. See also *Elements and Constants*, § 67, p. 131.

A, the mean of the moments around an axis perpendicular to that of rotation.

μ, the ratio of the mass of the moon to that of the earth.

N, the constant of nutation, as above defined.

P_1, that part of the luni-solar precession in a solar year of 365·2422 days which is due to the action of the moon.

P_1', that part of the same precession which is due to the action of the sun.

H, K, K', numerical coefficients, functions of the elements of the earth's orbit round the sun, the moon's orbit round the earth, and, in the case of K', of the volume of the earth and the intensity of its gravity.

We then have the following expressions for N, P_1, and P_1':

$$\left.\begin{aligned} N &= H \cos\epsilon \frac{\mu}{1+\mu} \frac{C-A}{C} \\ P_1 &= K \cos\epsilon \frac{\mu}{1+\mu} \frac{C-A}{C} \\ P_1' &= K' \cos\epsilon \frac{C-A}{C} \end{aligned}\right\} \qquad (33)$$

If we express N, P_1, and P_1' in seconds of arc, the logarithms of H, K, and K' as computed from gravitational theory are:

$$\left.\begin{aligned} \log H &= 5{\cdot}40289 \\ \log K &= 5{\cdot}975043 \\ \log K' &= 3{\cdot}72508 \end{aligned}\right\} \qquad (34)$$

The expression given by theory for the precessional constant of the preceding chapter is found by a comparison with (2) and (3). Since

$$P_1 + P_1' = p,$$

we have

$$p = \left(K \frac{\mu}{1+\mu} + K'\right) \frac{C-A}{C} \qquad (35)$$

If $\frac{C-A}{C}$, or the mechanical ellipticity of the earth, and μ, the mass of the moon, were known, the value of p could by means of the preceding equation be determined by theory, as could also that of N. But, as neither of these quantities has yet been determined with the precision necessary for this, it is more common to determine μ and $\frac{C-A}{C}$ from the equations (33) and (34) by means of the observed values of N and p.

Let us then take, as unknown quantities,

$$x=\frac{\mu}{1+\mu},$$
$$y=\frac{C-A}{A}.$$

We then have the equations

$$(Kx+K')\,y=54''\cdot 9066,$$
$$H\cos\epsilon xy=9''\cdot 21.$$

From the last equation we derive

$$\log xy=5\cdot 598\,84-10,$$
$$Kxy=37''\cdot 487,$$
$$K'y=17''\cdot 420,$$
$$y=0\cdot 003\,280\,57,$$
$$x=1\div 82\cdot 62,$$
$$\mu=1\div 81\cdot 62.$$

134. Numerical expression of the nutation.

When the theory of the motion of the pole is completely developed, it is found that the values of $\Delta\psi$ and $\Delta\epsilon$ are expressed as the sum of an infinite series of terms, each consisting of a coefficient into the sine or cosine of some combination of the following angles:

☊, the longitude of the moon's node;
L, the sun's mean longitude;
π', the longitude of the sun's perigee;
☾, the moon's mean longitude;
g, the moon's mean anomaly:
$D\equiv$ ☾ $-L$.

By referring to Appendix VII., it will be seen that several of the terms there given have coefficients of the order of magnitude $0''\cdot 01$, or less. These are commonly ignored as unimportant. The only terms in $\Delta\psi$ and $\Delta\epsilon$ used in practice are the following:

Terms depending on the moon's node.

$$\begin{aligned} \Delta\psi &= -17''\cdot 234 \sin \Omega, & \Delta\epsilon &= 9''\cdot 210 \cos \Omega, \\ &\quad - 0 \cdot 017 T \sin \Omega, & & \\ &\quad + 0 \cdot 209 \sin 2\Omega. & &\quad - 0''\cdot 090 \cos 2\Omega. \end{aligned}$$

Terms depending on the sun's longitude.

$$\begin{aligned} \Delta\psi &= -1''\cdot 272 \sin 2L, & \Delta\epsilon &= +0''\cdot 551 \cos 2L, \\ &\quad +0 \cdot 126 \sin (L - \pi'), & &\quad +0 \cdot 022 \cos (3L - \pi'), \\ &\quad -0 \cdot 050 \sin (3L - \pi'), & &\quad -0 \cdot 009 \cos (L + \pi'). \\ &\quad +0 \cdot 021 \sin (L + \pi'). & & \end{aligned}$$

Terms depending on the moon's longitude.

$$\begin{aligned} \Delta\psi &= -0''\cdot 204 \sin 2☾, & \Delta\epsilon &= +0''\cdot 089 \cos 2☾, \\ &\quad +0 \cdot 068 \sin g, & &\quad +0 \cdot 018 \cos (2☾ - \Omega), \\ &\quad -0 \cdot 034 \sin (2☾ - \Omega), & &\quad -0 \cdot 005 \cos (2☾ - g), \\ &\quad -0 \cdot 026 \sin (2☾ + g), & & \\ &\quad +0 \cdot 011 \sin (2☾ - g), & &\quad +0 \cdot 011 \cos (2☾ + g). \\ &\quad +0 \cdot 015 \sin (2D - g), & & \\ &\quad +0 \cdot 006 \sin 2D. & & \end{aligned}$$

The terms dependent on the sun's longitude are sometimes modified by using the true longitude instead of the mean as the argument, thus assimilating them with the aberration.

Putting

$$\odot, \text{ the sun's true longitude,}$$

we have, to terms of the first order in the eccentricity,

$$\begin{aligned} L &= \odot - 2e \sin (\odot - \pi'), \\ 2L &= 2\odot - 4e \sin (\odot - \pi'), \\ \sin 2L &= \sin 2\odot - 4e \sin (\odot - \pi') \cos 2\odot, \\ &= \sin 2\odot - 2e \sin (3\odot - \pi') + 2e \sin (\odot + \pi'), \\ \cos 2L &= \cos 2\odot - 2e \cos (3\odot - \pi') + 2e \cos (\odot + \pi'). \end{aligned}$$

In all the terms except the first we may use $\odot$ for L, the error thus arising being only $0''\cdot 002$. We have

$$2e = \cdot 0335.$$

Multiplying the above value of $\sin 2L$ by $1''\cdot272$, and combining the smaller terms of the product with the corresponding ones of the original expression, we find, for the sun-terms,

$$\begin{aligned}\Delta\psi = &-1''\cdot272 \sin 2\odot,\\ &+0\ \cdot126 \sin(\odot - \pi'),\\ &-0\ \cdot007 \sin(3\odot - \pi'),\\ &-0\ \cdot022 \sin(\odot + \pi').\end{aligned}$$

Substituting for π' its numerical value for 1900, which we may take as a constant,

$$\pi' = 281^\circ\cdot2,$$

and combining the second and fourth terms into one, we find

$$\begin{aligned}\Delta\psi = &-1''\cdot272 \sin 2\odot,\\ &+0\ \cdot147 \sin(\odot + 82^\circ),\\ &-0\ \cdot007 \sin(3\odot + 79).\end{aligned}$$

By a similar process we find, for the nutation of the obliquity,

$$\begin{aligned}\Delta\epsilon = &+0''\cdot551 \cos 2\odot,\\ &+0\ \cdot009 \cos(\ \odot - 79^\circ),\\ &+0\ \cdot004 \cos(3\odot + 79^\circ).\end{aligned}$$

NOTES AND REFERENCES TO PRECESSION AND NUTATION.

The fact of the precession of the equinoxes was originally discovered by Hipparchus, who flourished during the second century before the Christian era. His method was the same in principle as that adopted in our time for determining the position of the equinoxes among the stars. It was seen by the astronomers of ancient times, that there were two methods of determining the length of the year: the one by the interval between the times of the equinoxes; the other by the time of one apparent revolution of the sun among the stars. As, owing to the invisibility of the stars when the sun was above the horizon, it was impossible to compare the sun and stars directly, an indirect method was adopted, using the position of the moon as an intermediate point of reference. At the middle of a total eclipse of the moon, the latter was known to be directly opposite the sun, and its position among the stars could be determined. The distance of the moon from the sun could also be measured before sunset, and from a star after sunset.

The times of the equinoxes were those when sunset was exactly opposite to sunrise. By comparing the results of its own observations with those of his predecessors, it was found by Hipparchus that the position of the stars

Spica and Regulus had changed relative to the equinoxes, and that, while the solar year as determined by the equinoxes was several minutes less than 365¼ days, the time of revolution among the stars was several minutes greater. His estimate of the motion was, however, materially too small, being 1° in a hundred years instead of 1° in 70 years, which we now know to be the case.

In our time the precessional motion is determined in two ways: from the observed declinations of the stars and from their observed R.A.'s. Owing to the continual motion of the pole toward the position of the equinox, the stars situated in less than 90° of R.A. on either side of the equinoxes are continually increasing in declination; while those in the opposite quarter of the heavens are diminishing. Hence by comparing the observed declinations of the stars at epochs as distant as possible, the amount of the annual motion of the pole, or the value of the speed n, can be determined.

The actual R.A.'s of the stars being determined by comparison with the sun on a system which will be explained in a subsequent chapter, the actual motion of the equinoxes along the equator, or the value of m, can also be determined by observation. The actual amount of the motion is inferred from a combination of the two methods, taking the means which seem most probable when due allowance is made for the various sources of error to which each is subject.

That component of the motion of the ecliptic which takes place around the line of the equinoxes as an axis, or the value of $\kappa \cos N$, can be determined by observations of the obliquity at various epochs, but the component at right angles to this, on which alone the motion of the equinox depends, and which enters through the quantity λ, cannot be accurately determined by direct observation.

The motion of the ecliptic can also be determined theoretically from the action of the planets, when the masses of the latter are known. Our knowledge of these motions is subject to correction, and, in consequence, the numbers expressing the motion have also been corrected from time to time. Two independent corrections, one to the motion of the equator and the other to that of the ecliptic, have been made from time to time. This has been productive of more or less confusion in the combination of the two numbers so as to obtain the definitive values of the precessional motion.

The first values of these motions which have been extensively used in astronomy were derived by Bessel, and are found in a memoir to which a prize was awarded by the Berlin Academy of Sciences in 1815. These values were afterward corrected by Bessel and the results embodied in the *Tabulae Regiomontanae*, long the most accurate handbook of formulae, constants, and tables relating to the positions of the stars. The values here found were, therefore, in very general use for a number of years.

When the Poulkova Observatory was founded (1839), one of the objects mainly in view was the accurate determination of the positions of the principal fixed stars, and of the constants pertaining to their reduction. This led to an investigation of the constant of precession by Otto Struve, which was published by the St. Petersburg Academy of Science in 1843, and was based upon a comparison of the observations of Bradley, 1750-1755, with those of Bessel. The motions found by Struve were modified by Peters, and are markedly larger than those of Bessel. They gradually replaced the latter, their use becoming general through the last half of the 19th century.

Values were also derived by Leverrier and slightly modified by Oppolzer. The following is a list of the motions just mentioned:

Annual precessional motions by various authorities for the epoch 1850.

	General Precession.	*m*.	*n*.
Bessel - - -	50″·2357	46″·0591	20″·0547
Struve-Peters - -	50 ·2522	46 ·0764	20 ·0564
Leverrier* - -	50 ·2357	46 ·0601	20 ·0524
Oppolzer - - -	50 ·2346	46 ·0593	20 ·0515

Values of the precession have also been derived by LUDWIG STRUVE, BOLTE, DREYER (in the journal *Copernicus*, vol. ii., Dublin, 1882), and NYRÉN.

PETERS, C. A. F., *Numerus Constans Nutationis*, published by the St. Petersburg Academy of Sciences, contains the first thorough development of the theory of Nutation.

OPPOLZER, *Lehrbuch zur Bahnbestimmung*, Leipzig, 1882, contains an extended mathematical theory of precession and nutation; his numerical values are based on those of Leverrier.

In 1896 a conference of the directors of the National Ephemerides of England, France, Germany, and the United States was held at Paris for the purpose of deciding upon a uniform set of astronomical constants, and a system of star-reductions to be adopted in the several publications. Values of the precessional motions were also worked out for this purpose during the following year, in a research found in *Astronomical Papers of the American Ephemeris*, vol. viii. At that time the reinvestigation of the motions and elements of the planets had just been completed, which led to a more rigorous determination of the motion of the ecliptic. The resulting precessional motions are those used in the present work, and mostly adopted in the national ephemerides since 1901.

The *Elements of the four inner Planets and the Fundamental Constants of Astronomy*, published by the American Nautical Almanac in 1895, also gives values of the precessional motions. But these were superseded by the value of 1897, which had not then been worked out.

* For the Julian, not the solar year.

The amount of the nutation, computed from formulae substantially identical with those just given, is found in all the national ephemerides for intervals admitting of convenient interpolation to any date. The periods of the terms depending on the moon's node, and on the sun's longitude, are so long that an ephemeris for every tenth day will suffice so far as they are concerned. Those depending on the moon's longitude, if used at all, must be given for every day. As for many astronomical purposes, it is sufficient and sometimes even desirable to omit the small terms in the nutation, the two classes of terms are given separately in the *American Ephemeris* and in the British *Nautical Almanac*. In the former the ephemeris of the larger terms is given for every ten days only; and that in two separate tables,—one containing the values computed from the old constants of Struve and Peters, the other from the present adopted values. A separate table is given of the small terms for every day. In the British Ephemeris both classes of terms are given for every day. In the *Connaissance des Temps* the complete nutation is given for every day. The reason for tabulating the two classes of terms separately will be set forth in the chapter on the reduction of apparent places of the fixed stars.

For convenience the following values of the term of the nutation depending on the longitude of the node are given.

	$\Delta\psi$.	$\Delta\epsilon$.
BESSEL, *Tabulae Regiomontanae*,	$-16''\cdot783 \sin \Omega$	$8''\cdot977 \cos \Omega$
PETERS, *Numerus constans*,	$-17\ \cdot258$	$9\ \cdot224$
OPPOLZER, *Bahnbestimmung*,	$-17\ \cdot274$	$9\ \cdot236$

PART III.

REDUCTION AND DETERMINATION OF POSITIONS OF THE FIXED STARS.

CHAPTER X.

REDUCTION OF MEAN PLACES OF THE FIXED STARS FROM ONE EPOCH TO ANOTHER.

135. The *mean place* of a fixed star at any time is its apparent position on the celestial sphere, as it would be seen by an observer at rest on the sun. It is commonly expressed by coordinates referred to the mean pole and equinox of the beginning of some year.

The *apparent place* of such a star is its position on the sphere as it is actually seen by an observer on the moving earth, referred to the actual pole and equinox of the date.

The problem of the reduction of places of the fixed stars is that of determining the apparent place at one time from the mean place at another, or *vice versa*. It involves correction for the effects of the following causes:

1. The proper motion of the star between the two epochs;
2. Precession;
3. Nutation;
4. Aberration;
5. Annual Parallax.

Of these causes 1, 4, and 5 change the actual direction in which the star is seen, while 2 and 3 change only the axes of coordinates, without affecting the actual direction of the star.

The reduction involves two steps. Firstly, the mean place of the star is reduced from one epoch to the other by applying the effects of precession and proper motion. Secondly, this mean place is corrected for the effect of nutation, aberration, and parallax.

The correction for parallax is generally neglected in ephemerides, owing to its minuteness, and the lack, except in a few cases, of exact knowledge of its value.

The reduction of the mean place from one epoch to another is so far distinct from the computation of the apparent place that we consider them in separate chapters. In the present chapter the theory of mean places, as affected by proper motion and precession, will be studied.

In practice mean places are always referred to the equator and equinox of the beginning of some solar year. Hence the reduction for precession is always made for an integral number of years.

Section I. The Proper Motions of the Stars.

136. Each fixed star is, with very few exceptions, so widely separated from all others that, during our time, and perhaps

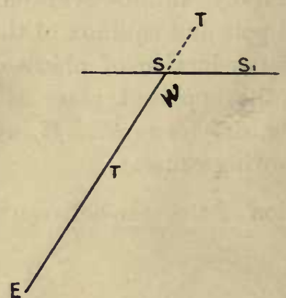

FIG. 29.

for ages to come, it may be supposed to move in a straight line with uniform velocity. In cases where this is not true, owing to the star being a member of a system, it is still true of the centre of mass of the system. What we have therefore to consider is the effect of the uniform rectilinear motion of a star in space upon its apparent motion on the celestial sphere.

Let E be the position of the earth, or more exactly that of the sun, this being the mean position of the earth during any one year, and therefore the actual origin to which the mean positions and motions of the stars are referred.

Let the star be moving relatively to the sun in the direction SS_1 with the uniform linear speed v, and let us put

r, the distance ES of the star, called also the *radial line*;

W, the angle ESS_1 which the direction of the motion makes with the radial line;

ρ, the speed with which r is increasing, called the *radial velocity*;

π, the annual parallax of the star;

μ, the angular speed of its apparent motion on the celestial sphere, as seen from the sun. We then have

$$\left.\begin{aligned} \mu &= \frac{v \sin W}{r} = \frac{dW}{dt} \\ \rho &= \frac{dr}{dt} = -v \cos W \end{aligned}\right\} \qquad \text{(1)}$$

We now have to find the variation of μ with the time.

By differentiation of the first equation and simple reductions, we find

$$D_t\mu = 2\frac{v^2}{r^2}\sin W \cos W.$$

Substituting for $\sin W$ and $\cos W$ their values from (1), this equation reduces to

$$D_t\mu = -\frac{2\mu\rho}{r}. \qquad \text{(2)}$$

Instead of r, the distance of the star, we use its annual parallax, connected with a, the mean distance of the earth from the sun, by the relation

$$a = r \sin \pi.$$

Using a as the astronomical unit of length, we should define v in terms of this unit, using it to indicate radii of the earth's orbit in unit of time. Doing this and substituting $\sin \pi$ for $\frac{1}{r}$ we find

$$D_t\mu = -2\mu\rho \sin \pi. \qquad \text{(2')}$$

In this equation μ is expressed in circular measure. In astronomical practice it is common to express μ in seconds of arc per year, and the radial velocity in kilometres per second. The value of ρ to be used in the last equation is found by dividing the speed in kilometres per second by 4·75.*

If we put μ'' for the annual motion in seconds of arc, and π'' for the parallax in seconds of arc, we shall have

$$\left.\begin{aligned} \mu &= \mu'' \sin 1'' \\ \sin \pi &= \pi'' \sin 1'' \\ D_t\mu'' &= -2\mu''\pi''\rho \sin 1'' \end{aligned}\right\} \quad \text{(3)}$$

The radial speed ρ can be measured only with the spectroscope, and is known only for a few hundred of the brighter stars. Among the stars whose radial speed and parallax have both been determined, 1830 Groombridge is that which will give much the largest value of this change. The measures of its radial speed at the Lick Observatory give

$$\rho = -20.$$

For it we have also

$$\mu'' = 7''$$

and, with much uncertainty,

$$\pi'' = 0''\!\cdot\!14,$$

and thus, for 1830 Groombridge,

$$D_t\mu'' = +0''\!\cdot\!00019.$$

This change is too small to be detected until accurate observations shall have extended through fully a century; and as it is exceptionally large, the consideration of the change in the case of the stars in general belongs to the astronomy of the future. In the present state of astronomy we may, therefore, assume that by its proper motion each star moves on a great circle with an invariable angular speed. We put

μ, this constant angular speed;

* This factor is connected with the solar parallax by the relation

$$\text{Factor} = \frac{[1\cdot 62003]}{\odot\text{'s par. in secs.}}.$$

The value as given therefore corresponds to par. $= 8''\!\cdot\!776$. For the value $8''\!\cdot\!80$, still in common use, we have 4·7375 for the divisor.

N, the angle which its direction makes with the meridian of the star, counted from North toward East. We then have, for the rates of change in R.A. and Dec.,

$$\left.\begin{array}{ll}\text{Proper motion in R.A.,} & \mu_\alpha = \mu \sin N \sec \delta \\ \text{Proper motion in Dec.,} & \mu_\delta = \mu \cos N\end{array}\right\} \quad \ldots\ldots\ldots(4)$$

137. Reduction for proper motion.

The mean place of a star at any epoch is not necessarily referred to the equator and equinox of that epoch. We may have occasion to refer it to the coordinate axes of any other epoch. It follows that the reduction for proper motion is quite distinct from that for precession. We therefore begin by finding the effect of proper motion when the axes of reference remain fixed. We put

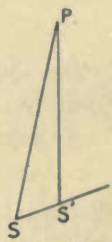

Fig. 30.

α_0, δ_0, N_0, the coordinates of the star and the direction of its proper motion at the initial epoch;

α, δ, N, the corresponding quantities for an epoch later by the time t, referred to the same equator and equinox.

In Fig. 30 let P be the pole. During the interval t the star will have moved over an arc

$$SS' = \mu t,$$

so that PSS' is a triangle in which

Angle $S = N_0$, for the initial date;
Exterior Angle $S' = N$, for the final date.

The equations between the parts of this triangle enable us to determine α, δ, and N by the rigorous equations

$$\begin{aligned}\cos\delta \sin(\alpha - \alpha_0) &= \sin N_0 \sin \mu t,\\ \cos\delta \cos(\alpha - \alpha_0) &= \cos\delta_0 \cos\mu t - \sin\delta_0 \cos N_0 \sin\mu t,\\ \sin\delta &= \sin\delta_0 \cos\mu t + \cos\delta_0 \cos N_0 \sin \mu t, \quad \ldots\ldots\ldots(5)\end{aligned}$$

$$\begin{aligned}\cos\delta \sin N &= \cos\delta_0 \sin N_0,\\ \cos\delta \cos N &= \cos\delta_0 \cos N_0 \cos\mu t - \sin\delta_0 \sin\mu t. \quad \ldots\ldots\ldots(6)\end{aligned}$$

The use of these equations in their rigorous form can be necessary only in the rare case, which has not yet occurred in practice, when the proper motion is an important fraction of the polar distance of the star. In all ordinary cases a development in powers of the time to t^2 will suffice, and even this last term will rarely be appreciable.

To effect this development we suppose SS' infinitesimal. We then have, from the triangle PSS',

$$\left.\begin{aligned} d\delta &= \mu \cos N\,dt \\ \cos\delta\,d\alpha &= \mu \sin N\,dt \\ dN = \sin\delta\,d\alpha &= \mu \sin N \tan\delta\,dt \end{aligned}\right\} \quad \text{..............(7)}$$

Then, by differentiating (4), substituting and reducing,

$$\left.\begin{aligned} (D_t\mu_\alpha) &= 2\mu_\alpha\mu_\delta \tan\delta \\ (D_t\mu_\delta) &= -\mu_\alpha^2 \sin\delta\cos\delta \end{aligned}\right\}. \quad \text{....................(8)}$$

The derivatives are enclosed in parentheses to distinguish them from the total variations when precession is included. Their value should satisfy the condition that the derivative of

$$\mu^2 = \mu_\alpha^2 \cos^2\delta + \mu_\delta^2$$

shall vanish, which we find to be the case by differentiating and substituting from (8) and (4).

The preceding equations presuppose that μ is expressed in circular measure. To transform them into the usual notation of seconds of arc, we must, in one of the factors of the second member, replace μ by $\mu \sin 1''$. Then

$$\left.\begin{aligned} (D_t\mu_\alpha) &= 2\mu_\alpha\mu_\delta \sin 1'' \tan\delta \\ (D_t\mu_\delta) &= -\mu_\alpha^2 \sin 1'' \sin\delta\cos\delta \end{aligned}\right\}, \quad \text{.................(8')}$$

which holds true when μ_α and μ_δ are expressed in seconds of arc.

By Taylor's theorem α and δ are expressed in the form

$$\left.\begin{aligned} \alpha &= \alpha_0 + \mu_\alpha t + \tfrac{1}{2}(D_t\mu_\alpha)t^2 \\ \delta &= \delta_0 + \mu_\delta t + \tfrac{1}{2}(D_t\mu_\delta)t^2 \end{aligned}\right\}, \quad \text{....................(9)}$$

where the quantities in the second member are the values for the initial epoch.

Putting $\Delta_1\alpha = \mu_\alpha t$ and $\Delta_1\delta = \mu_\delta t$, these quantities will be the principal terms of $\alpha - \alpha_0$ and $\delta - \delta_0$ respectively. Comparing with (8'), we see that the last terms of (9) may be written

$$\tfrac{1}{2}(D_t\mu_\alpha)t^2 = \Delta_1\alpha\Delta_1\delta \sin 1'' \tan\delta_0,$$
$$\tfrac{1}{2}(D_t\mu_\delta)t^2 = -\tfrac{1}{2}\Delta_1\alpha^2 \sin 1'' \sin\delta_0 \cos\delta_0.$$

It follows that we may write the reductions for proper motion in the form

$$\left.\begin{aligned}\alpha-\alpha_0&=\mu_\alpha t(1+\Delta_1\delta\tan\delta_0\sin 1'')\\ \delta-\delta_0&=\mu_\delta t-\tfrac{1}{2}\Delta_1\alpha^2\sin\delta_0\cos\delta_0\sin 1''\end{aligned}\right\}, \quad\text{.............(10)}$$

where $\Delta_1\alpha$ and $\Delta_1\delta$ are the values of $\mu_\alpha t$ and $\mu_\delta t$ expressed in seconds of arc. It is only when these quantities are exceptionally large that the last terms will become sensible. The equations (10) have the advantage of enabling us to determine, almost at a glance, whether the terms in t^2 are sensible. If they are, and if we require the proper motion as well as the position for the second epoch, the reduction may be made by using the mean value of the motion for the two epochs. We then begin by computing (8), and then the changes in the values of μ_α and μ_δ from the equations

$$\left.\begin{aligned}\Delta\mu_\alpha&=(D_t\mu_\alpha)t\\ \Delta\mu_\delta&=(D_t\mu_\delta)t\end{aligned}\right\}, \quad\text{.............................(11)}$$

which, being applied to the proper motions for the initial epoch, will give them for the final epoch, referred to the same equinox in the two cases. We then make the reduction of the position by the formulae

$$\left.\begin{aligned}\alpha-\alpha_0&=(\mu_\alpha+\tfrac{1}{2}\Delta\mu_\alpha)t\\ \delta-\delta_0&=(\mu_\delta+\tfrac{1}{2}\Delta\mu_\delta)t\end{aligned}\right\}. \quad\text{.....................(12)}$$

The reduction (12) is for proper motion alone, the axes of reference remaining fixed.

Section II. Trigonometric Reduction for Precession.

138. Rigorous formulae of reduction.

The problem before us now is: Having given the coordinates and proper motion of a star, referred to the equator and equinox of some initial date, to find the values of these quantities referred to the equator and equinox of some other date, the absolute position and rate of motion remaining unchanged.

There are two ways of solving this problem: one by a rigorous trigonometric computation; the other by a development in the powers of the time. We begin with the first.

Let P_0 and P (Fig. 31) be the positions of the poles at the two epochs, and S that of the star. Draw the arcs P_0E_0, and PE from the poles toward the respective equinoxes. Comparing this figure with Figure 26, we see that

$$\text{Angle } PP_0E_0 = \zeta_0,$$
$$\text{Angle } P_0PE = 180° - z,$$

where ζ_0 and z are the angles whose values have been developed in Chapter IX., §§ 130, 131.

We have also

$$\text{Angle } SP_0E_0 = \text{R.A. of star at initial epoch;}$$
$$\text{Angle } SPE = \text{R.A. at terminal epoch.}$$

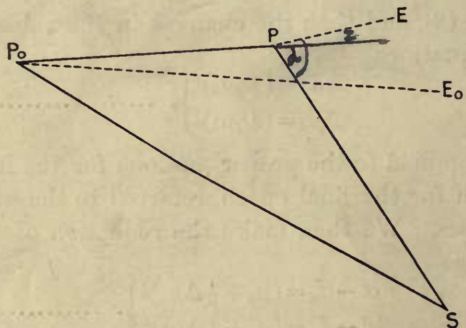

FIG. 31.

Let α_0, δ_0, $\mu_\alpha{}^0$, and $\mu_\delta{}^0$ be the coordinates and proper motions of the star when referred to the initial equinox; and

α, δ, μ^α, and μ_δ the same quantities referred to the final equinox.

In the triangle SPP_0 we have

$$\left.\begin{aligned} &\text{Angle at } P_0 = \alpha_0 + \zeta_0 \equiv a \\ &\text{Angle at } P = 180° - (\alpha - z) \equiv 180° - a' \\ &\text{Side} \quad P_0P = \theta \\ &\text{Side} \quad P_0S = 90° - \delta_0 \\ &\text{Side} \quad PS = 90° - \delta \end{aligned}\right\} \quad \ldots\ldots\ldots(13)$$

Of these parts the second and last are determined from the other three given parts. They may be found by the usual trigonometric equations

$$\left.\begin{aligned}\cos\delta\sin a' &= \cos\delta_0\sin a\\ \cos\delta\cos a' &= \cos\theta\cos\delta_0\cos a - \sin\theta\sin\delta_0\\ \sin\delta &= \sin\theta\cos\delta_0\cos a + \cos\theta\sin\delta_0\end{aligned}\right\}\quad\text{.........(14)}$$

These equations give the shortest computation so far as the number of quantities to be used is concerned. They may be adopted in the case of a star very near the pole. But their use requires 7-place logarithms; and in all ordinary cases they may, owing to the smallness of θ, be advantageously transformed into others which will give $a'-a$ and $\delta'-\delta$ in terms of the given data, and will require fewer figures in the logarithms. To do this we multiply the first by $\cos a$ and the second by $\sin a$ and subtract. Then we multiply the first by $\sin a$ and the second by $\cos a$ and add. The quotient of the two results gives us an equation which, after dividing both terms of the fraction by $\cos\delta$, we may write

$$\tan(a'-a) = \frac{N}{D}, \quad\text{..........................(15)}$$

where $N = (\cos a - \cos\theta\cos a + \sin\theta\tan\delta_0)\sin a,$
$D = \cos\theta\cos^2 a + \sin^2 a - \sin\theta\cos a\tan\delta_0.$

The coefficient of $\sin a$ in the numerator readily reduces to

$$\sin\theta\tan\delta_0 + 2\sin^2\tfrac{1}{2}\theta\cos a \equiv p, \quad\text{..................(16)}$$

and the denominator reduces to

$$1 - p\cos a.$$

Having determined $a'-a$ from (15), we see from the first two equations (13) that we have the following computations for the total change in R.A.:

$$\left.\begin{aligned}a &= \alpha_0 + \zeta_0\\ p &= \sin\theta(\tan\delta_0 + \tan\tfrac{1}{2}\theta\cos a)\\ \tan(a'-a) &= \frac{p\sin a}{1-p\cos a}\end{aligned}\right\},\quad\text{............(17)}$$

$$\left.\begin{aligned}\alpha &= a' + z = \alpha_0 + (a'-a) + \mathrm{m}\\ \alpha - \alpha_0 &= a' - a + \mathrm{m}\\ \text{where we put}\quad \mathrm{m} &= \zeta_0 + z.\end{aligned}\right\}\quad\text{.............(18)}$$

The reduction in declination is equal to the difference of the sides P_0S and PS, for which Napier's analogy, or the quotient of two of the Gaussian equations for the spherical triangle, gives us

$$\tan\tfrac{1}{2}(\delta-\delta_0)=\frac{\cos\frac{1}{2}(a'+a)}{\cos\frac{1}{2}(a'-a)}\tan\tfrac{1}{2}\theta. \quad\text{..........(19)}$$

139. Geometric signification of the constants.

The geometric signification of the arcs $a'-a$, ζ_0 and z, whose sum make up the reduction, can be more readily seen by a study of their relation to the equator than by the construction we have used. Let E_0Q and EQ be the two equators, inter-

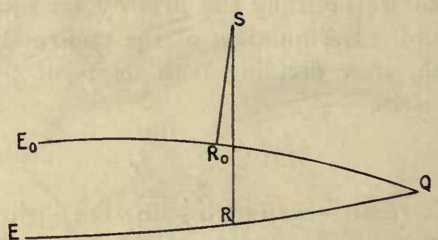

FIG. 32.

secting at Q, and E_0 and E the two equinoxes. We then see, by transferring the measures of the angles ζ_0 and z from their vertices at the poles to the equator, that

$$\zeta_0=90^\circ-E_0Q,$$
$$z=EQ-90^\circ.$$

Then since $\alpha_0=E_0R_0$ and $\alpha=ER$, we have

$$\alpha-\alpha_0=\zeta_0+z+R_0Q-RQ,$$
$$a'-a=R_0Q-RQ.$$

140. Approximate formulae.

The preceding rigorous formulae are necessary only in the case of stars near the pole. In all ordinary cases the reduction may be simplified by the following process. We write p in the form

$$p=p_0+\Delta p,$$

where

$$\left.\begin{aligned}p_0&=\sin\theta\tan\delta_0\\ \Delta p&=2\sin^2\tfrac{1}{2}\theta\cos a\end{aligned}\right\}\quad\text{..........(20)}$$

Now develop $\tan(a'-a)$ in (17) in powers of Δp.

$$\tan(a'-a) = \frac{p_0 \sin a}{1 - p_0 \cos a} + \frac{\sin^2 \frac{1}{2}\theta \sin 2a}{(1 - p_0 \cos a)^2} + \text{etc.} \quad \text{.........(21)}$$

To estimate the value of the second term we note that, in a reduction extending over a hundred years, we have, approximately,

$$\tfrac{1}{2}\theta = 1002'' = 0{\cdot}005,$$

whence

$$\sin^2 \tfrac{1}{2}\theta = 5''{\cdot}0 = 0^s{\cdot}33.$$

This is the maximum value of the numerator of the last term of (21) for this particular case. Since p_0 is small, unless the star is near the pole, the denominator will generally differ little from unity. For a reduction through 100 years approximate values of p_0 or p are

$$\text{Dec.} = 80°; \quad p = 0{\cdot}057.$$
$$\text{,,} \quad = 85°; \quad p = 0{\cdot}115.$$

The equation (21) will, therefore, suffice in all cases when the star is not very near the pole. Its computation may be facilitated by dividing $\tan(a'-a)$ into three parts, using the notation

$$\left.\begin{aligned} \Delta_0 a &= \frac{p_0 \sin a}{1 - p_0 \cos a} \\ \Delta_1 a &= \sin^2 \tfrac{1}{2}\theta \sin 2a \\ \Delta_2 a &= \left(\frac{1}{(1 - p_0 \cos a)^2} - 1\right) \Delta_1 a \equiv F \Delta_1 a \end{aligned}\right\} \quad \text{..........(22)}$$

Then

$$a' - a = \Delta_0 a + \Delta_1 a + \Delta_2 a - \text{Red. from arc to tangent} \equiv \Delta a. \quad \text{...(23)}$$

141. Construction of tables for the reduction.

The computation of these quantities is shortened by the tables of Appendix IV., of which the construction is this:

We express the four parts of a in seconds of time by dividing them by $\sin 1\text{ s.} = 15 \sin 1''$, the reciprocal of which we call h, so that

$$\log h = 4{\cdot}138\,334.$$

When, and only when, necessary to avoid confusion, we indicate this form of expression by a suffix s, so that p_s means p expressed in seconds of time, or $p_s = hp$. We then have from (22)

$$\Delta_0 a_s = \frac{p_{0,s} \sin a}{1 - p_0 \cos a},$$

with similar expressions formed by multiplying $\Delta_1 a$ and $\Delta_2 a$ by p. We then have

$$\alpha = \alpha_0 + \Delta a_s + \mathrm{m} \quad \text{......................(24)}$$

The constants and formulae for all the cases which ordinarily occur are found in Appendix IV., which also contains tables to facilitate the reduction. Table XII. of this Appendix gives the logarithm of

$$K = \frac{1}{1 - p \cos a}$$

for usual values of p, the computation of $\log K$ being made with $p \cos a$ in circular measure, but the argument being multiplied by the factor h, so as to be expressed in seconds of time.

Table XIII. gives the value of $\Delta_1 a$, the argument θ being replaced by the time elapsed between the two epochs, to which it is nearly proportional.

Table XIV. gives the factor F, by which $\Delta_1 a$ is multiplied to find $\Delta_2 a$.

Table XV. gives the reduction from the sum $\Delta_0 a + \Delta_1 a + \Delta_2 a$, (which is the tangent of Δa expressed in seconds of time) to Δa itself. It is always subtractive numerically.

142. Reduction of the declination.

Unless the motion of the pole is an important fraction of the polar distance of the star, we may use, instead of (19), the approximate equation

$$\delta = \delta_0 + \theta \cos (a + \tfrac{1}{2}\Delta a) \sec \tfrac{1}{2}\Delta a. \quad \text{..............(25)}$$

143. Failure of the approximation near the pole.

The boundary of the region within which the use of Δp ceases to be convenient is approximately a spherical lemniscate having the pole as centre, and the meridian through 0 h. and 12 h. of R.A. as its axis. Practically we may replace this curve by a pair of circles as shown in Figure 33.

The length of the semi-axis a may be taken as 1° for every 10 years of the interval through which the reduction extends. The limits are, in general, given by the equation

$$\text{Polar Distance} = 0°{\cdot}10t \cos \alpha.$$

The argument of Table XV. approaches the tabular limit when

$$\text{Polar Distance} = 0°{\cdot}04t \sin \alpha.$$

The corresponding limiting curve is a lemniscate similar to that just defined, but having its axis at right angles to that of the

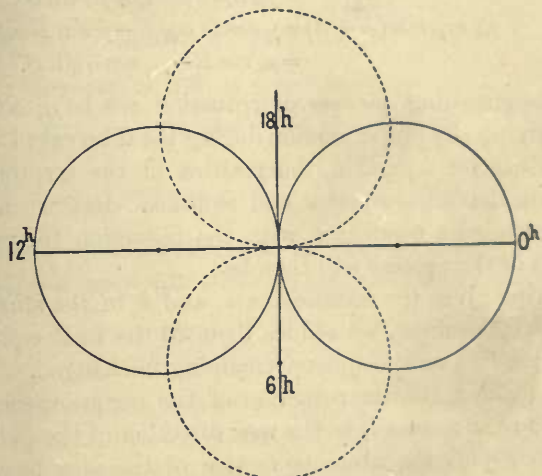

Fig. 33.

other. It is shown by the two dotted circles. If, owing to the position of the star being within the limits just defined, or to any other reason, the rigorous formulae (17) are used, the computation can still be facilitated by using the table for K.

144. Reduction of the proper motion.

The proper motion of the star when referred to the final equinox will also be different from that referred to the initial equinox, owing to the change in the direction of the hour circles. To reduce it to the final equinox, let us again refer to the triangle P_0PS, formed by the two poles and the star. The angle N of § 136, Eq. (4) will be changed by the angle S, so that, putting

N', the value of N, referred to the final equinox, we shall have

$$N' = N + S.$$

The angle S may be computed by the equation

$$\sin S = \sin\theta \sin a \sec\delta. \quad \text{.....................(26)}$$

The proper motions referred to the final equinox will then be given by the equations

$$\left.\begin{aligned} \mu_\alpha' \cos\delta' = \mu \sin(N+S) &= \mu \sin N \cos S + \mu \cos N \sin S \\ &= \mu_\alpha \cos\delta \cos S + \mu_\delta \sin S \\ \mu_\delta' = \mu \cos(N+S) &= \mu \cos N \cos S - \mu \sin N \sin S \\ &= \mu_\delta \cos S - \mu_\alpha \cos\delta \sin S. \end{aligned}\right\} \quad (27)$$

In the preceding process of reduction we have commenced with applying the proper motion during the interval of reduction so as to use for α_0 and δ_0 the position at the terminal epoch, referred to the initial equator and equinox. But we may with equal convenience commence with the reduction for precession. The steps of the process will then be:

1. Having given the coordinates α_0 and δ_0 of the star referred to the initial equinox, we reduce them to the final equinox, the absolute position on the sphere remaining unchanged.

2. We make a similar reduction of the instantaneous proper motion, so as to reduce it to the new direction of the pole.

3. We compute the absolute motion of the star between the two epochs by reducing the position obtained by step 1 with the proper motion obtained by step 2.

As an example of the reduction, we take the star 1830 Groombridge, of which the position and centennial proper motion for the date 1875·0 are:

$$\left.\begin{aligned} \alpha &= 11\text{ h. } 45\text{ m. } 46{\cdot}120\text{ s.}; & \mu_\alpha &= +34{\cdot}198\text{ s.} \\ \alpha &= 176^\circ\ 26'{\cdot}53; & &= 512''{\cdot}97 \\ \delta &= +38^\circ\ 36'\ 55''{\cdot}55; & \mu_\delta &= -577''{\cdot}97 \end{aligned}\right\} \ldots \text{(A)}$$

Assuming that a 5-place table of sines and cosines to time is not at hand, we have reduced α to arc.

We call this position (A), and we propose to reduce it to 1910·0, an interval of 35 years.

We begin by computing the absolute motion of the star between the two epochs, supposing no change of the equinox of reference. We first require the change in proper motion, which may be computed by (8′):

$\log \mu_\alpha$	2·7101	$\log \mu_\alpha{}^2$	5·4202
,, μ_δ	2·7619 n	,, sin 1″	4·6856 − 10
,, 2 sin 1″	4·9866 − 10	,, $\sin\delta$	9·7952 − 10
,, $\tan\delta$	9·9024 − 10	,, $\cos\delta$	9·8928 − 10
,, $D_t\mu_\alpha$	0·3610 n	,, $D_t\delta$	9·7938 − 10

$$D_t\mu_\alpha = -2''\cdot296 = -0\cdot153 \text{ s.},$$
$$D_t\mu_\delta = -0''\cdot622.$$

Hence in 35 years, or 0·35 of a century,

$$(\Delta\mu_\alpha) = -0\cdot054 \text{ s.}; \quad (\Delta\mu_\delta) = -0''\cdot22,$$

and, for 1910,

$$\mu_\alpha = 34\cdot144 \text{ s.}; \qquad \mu_\delta = -578''\cdot19, \quad \text{.........(B)}$$

which motions are still referred to the original axes.

Reducing the position to 1910 by (10) or (11), we now find

$$\left.\begin{aligned} \alpha &= \ \ 176° \ 29' \ 31''\cdot20 \\ \delta &= + \ \ 38 \ \ 33 \ \ 33 \ \cdot22 \end{aligned}\right\}. \quad \text{.....................(B)}$$

This second position, which we call (B), is that of the star in 1910, when referred to the equinox of 1875.

We thus have two positions of the star—the one for 1875, the other for 1910—both referred to the equator and equinox of 1875. We shall now reduce both of these positions to the equator and equinox of 1910.

We find the following constants of reduction from the general expressions of Appendix IV., making the reversal there explained:

$$\zeta_0 = 13' \ 26''\cdot50 = 13'\cdot44,$$
$$\log h \sin\theta = 1\cdot670\,04,$$
$$\text{m} = 1 \text{ m. } 47\cdot527 \text{ s.}$$
$$= 26' \ 52''\cdot90,$$
$$\theta = 11' \ 41''\cdot66 = 701''\cdot66.$$

The computation is now the following, starting in (A) with the position in 1875, and in (B) with that for 1910:

Reductions of 1830 Groombridge from the equinox and equator of 1875 to those of 1910.

	A.	B.
R.A., initial equinox, α_0	176° 26′·53	176° 29′·52
ζ_0, 1875 to 1900	13·44	13·44
a	176° 39′·97	176° 42′·96
$\log \tan \delta_0$	9·902 40	9·901 52
$\log h \sin \theta$	1·670 04	1·670 04
$\log \cos a$	9·999 26n	9·999 29n
$\log p_s \cos a$	1·571 70n	1·570 85n
$\log K$ Tab.	− 0·001 18	− 0·001 17
$\log \sin a$	8·764 57	8·758 04
$\log p_s$	1·572 44	1·571 56
$\log \Delta_0 a$	0·335 83	0·328 43
$\Delta_0 a$	2·167	2·130
$\Delta_1 a$	− ·004	− ·004
	h. m. s.	h. m. s.
m	0 1 47·527	0 1 47·527
α_0(Eq. 1875)	11 45 46·120	11 45 58·080
α(Eq. 1910)	11 47 35·810	11 47 47·733
$a + \frac{1}{2}\Delta a$	176° 40′·24	176° 43′·22
$\cos(a + \frac{1}{2}\Delta a)$	9·999 26n	9·999 29n
$\log \theta$	2·846 12	2·846 12
$\sec \frac{1}{2}\Delta a$	0	0
$\log \Delta\delta$	2·845 38n	2·845 41n
$\Delta\delta$	= − 700″·45	− 700″·50
δ_0(Eq. 1875)	38° 36′ 55″·55	38° 33′ 33″·22
δ(Eq. 1910)	38 25 15 ·10	38 21 52 ·72

The next step is to make the corresponding reduction of the proper motion from the one equinox to the other. We find from (26)

$$\text{nat. } \sin S = +0{\cdot}000\,25.$$
$$\text{nat. } \cos S = 1{\cdot}000\,00.$$

The two computations then are:

	A.	B.
$\log \mu_\alpha$	2·710 09	2·709 41
$\log \cos \delta_0$	9·892 85	9·893 18
$\log \mu_\alpha \cos \delta_0$	2·602 94	2·602 59
$\mu_\alpha \cos \delta_0$	400″·81	400″·49
$\mu_\delta \sin S$	−0 ·14	−0 ·14
$\mu_\alpha' \cos \delta$	400·67	400·35
log ,, ,,	2·602 79	2·602 44
$\log \sec \delta$	0·105 98	0·105 64
$\log \mu_\alpha'$	2·708 77	2·708 08
μ_α' (Eq. 1910)	511″·41	510″·60
	= 34·094 s.	34·040 s.
$\mu_\delta \cos S = \mu_\delta$	−577·97	−578·19
$-\mu_\alpha \cos \delta \sin S$	− 0·10	− 0·10
μ_δ' (Eq. 1910)	−578·07	−578·29

The results (B) are the final ones. But results (A) are not because they give the position and motion of the star on the sphere at the initial epoch, referred, however, to the equinox of 1910. We therefore continue the computation (A) by finding the change in α, δ, and μ, due to the proper motion of the star on the sphere during the interval between the epochs.

Beginning with the proper motions, we shall find the centennial variations, and hence the reductions to be almost the same as in the first computation. Thus we have for the centennial proper motions for epoch 1875, and equinox of 1910:

Computation (A), -	$\mu_\alpha' = 34·094$ s.	$\mu_\delta' = -578''·07$
Change in 35 years, - -	− 0·054	− 0 ·22
Motions for 1910, - -	34·040	−578 ·29
Results of Computation B	34·040	−578 ·29

The two results should in theory be the same.

Next, to reduce position (B), the mean of the proper motions (B) for the two epochs are found to be:

$$\mu_\alpha = 34^s·067\,;\ \mu_\delta = -578''·18.$$

These, multiplied by 0·35, give for the reductions from 1875 to 1910, $\Delta\alpha = +11·923$ s.; $\Delta\delta = -3'\ 22''·36$.

These, applied to the results of computation (A), namely:

		h.	m.	s.				
	$\alpha=$	11	47	35·810;	$\delta=$	38°	25′	15″·10,
give	$\alpha=$	11	47	47·733;	$\delta=$	38	21	52 ·74.
(B) gave				47·733				52 ·72.
Discrepancy				·000				·02.

Reduction of β Ursae Minoris from 1755 *to* 1875.

As a second example, we reduce the position of β Ursae Minoris for 1755, as found in Auwers' catalogue of Bradley stars, from that epoch to 1875, an interval of 120 y. Omitting proper motion, we give the computation both by the rigorous formulae and the development in powers of Δp. The position for 1755 is:

$$\alpha_0 = 14 \text{ h. } 51 \text{ m. } 42\text{·}56 \text{ s.} = 222° \; 55'\text{·}64,$$
$$\delta_0 = +75° \; 9' \; 23''\text{·}2.$$

The constants of reduction are:

$$\zeta_0 = 46'\text{·}05, \quad m = 6 \text{ m. } 8\text{·}495 \text{ s.},$$
$$\log h \sin\theta = 2\text{·}205\,27, \quad \theta = 40' \; 6''\text{·}42.$$

Whence $a = 223° \; 41'\text{·}69.$

The two computations are:

Rigorous.

$\tan\frac{1}{2}\theta$	7·766
$\cos a$	9·859n
$\tan\frac{1}{2}\theta\cos a$	7·625n
$\log\tan\delta_0$	0·576 713
Diff.	2·952
Subt. log (Tab. B)	− 486
	0·576 227
$\sin\theta$	8·066 937
p	8·643 164
$\sin a$	9·839 363n
$p\sin a$	8·482 527n
$\cos a$	9·859 16n
$p\cos a$	8·502 32n
$1-p\cos a$	0·013 592
$\tan\Delta a$	8·468 935
Δa −	1° 41′ 10″·64

Approximate.

$\log\tan\delta_0$	0·576 72
$\log h\sin\theta$	2·205 27
$\log p_s$	2·781 99
$\log\cos a$	9·859 16n
Arg.	2·641 15n
$\log K$ (Tab. I.)	− ·013 61
$\log\sin a$	9·839 36n
$\log\Delta_0 a$	2·607 74n
$\Delta_0 a$	− 405·26
$\Delta_1 a$ (Tab. II.)	+ 0·459
$F\Delta_1 a$ (Tab. III.)	− 0·028
Red. (Tab. IV.)	+ 0·117
Δa	− 404·712

Rigorous.	h.	m.	s.	Approximate.	h.	m.	s.
$\Delta a =$	-0	6	44·709	$\Delta a =$	-0	6	44·712
m	$+0$	6	8·495	m	$+0$	6	8·495
α_0	14	51	42·56	α_0	14	51	42·56
(Eq. 1875) α	14	51	6·35	α	14	51	6·34

Declination.		Declination.	
$\frac{1}{2}(a'+a)$	222° 51′·10	$\frac{1}{2}\Delta a$	−0° 50′·59
$\frac{1}{2}(a'-a)$	−0 50·59	$a+\frac{1}{2}\Delta a$	222 51·10
$\tan\frac{1}{2}\theta$	7·765 921	$\log\theta$	3·381 372
$\cos\frac{1}{2}(a'+a)$	9·865 173n	$\cos(a+\frac{1}{2}\Delta a)$	9·865 173n
$\sec\frac{1}{2}(a'-a)$	47	$\sec\frac{1}{2}\Delta a$	47
$\tan\frac{1}{2}(\delta-\delta_0)$	7·631 141n	$\log\Delta\delta$	3·246 592n
$\frac{1}{2}(\delta-\delta_0)$	−0° 14′ 42″·19	$\Delta\delta$	−1764″·38
$\delta-\delta_0$	−0 29 24·38		= −0° 29′ 24″·38
δ_0	75 9 23·20	δ_0	75 9 23·20
(Eq. 1875) δ	74 39 58·82	δ	74 39 58·82

Reduction of Polaris to 2100.

As a case where the original form (14) of the equations of reduction is most convenient, let us reduce the mean place of the pole-star to 2100, an epoch two years before the nearest approach of the pole to the star. The position and centennial proper motion for 1900 are:

$$\alpha = 1\text{ h. }22\text{ m. }33·19\text{ s.},\qquad \mu_\alpha = +13·64\text{ s.},$$
$$\delta = 88°\ 46'\ 26''·61,\qquad \mu_\delta = +0''·33.$$

We first make the reduction for proper motion during the interval of 200 years.

$$\Delta_1\alpha = 2·00 \times \mu_\alpha = +27·28\text{ s.},$$
$$\Delta_1\delta = 2·00 \times \mu_\delta = +0''·66.$$

In the equation (10) the last terms will be negligible. We therefore have, for the position of (2100) referred to the origin of 1900,

$$\alpha_0 = 1\text{ h. }23\text{ m. }0·47\text{ s.},$$
$$\delta_0 = 88°\ 46'\ 27''·27.$$

The constants are (App. IV.):

$$\zeta_0 = 1^\circ\ 16'\ 49''\cdot 8,$$
$$z = 1\ \ 16\ \ 53\ \cdot 0,$$
$$\theta = 1\ \ \ 6\ \ 47\ \cdot 3.$$

α_0	20° 45 7″·05	$\cos\theta\cos\delta_0\cos a$	8·297 233 5
ζ_0	1 16 49 ·8	$\sin\theta\sin\delta_0$	8·288 300 0
a	22 1 56 ·85	diff.	0·008 933 5
		subt. log	1·682 290 0
$\sin a$	9·574 183 9	$\cos\delta\cos a'$	6·606 010 0
$\cos\delta_0$	8·330 249 1	$\cos\delta\sin a'$	7·904 433 0
$\cos a$	9·967 066 4	$\tan a'$	1·298 423 0
$\sin\theta$	8·288 399 4	a'	87° 7′ 13″·40
$\sin\delta_0$	9·999 900 6	z	1 16 53 ·0
$\cos\theta$	9·999 918 0	α(Eq. 2100)	88 24 6 ·4
			= 5 h. 53 m. 36·43 s.
		$\sin a'$	9·999 451 2
		$\cos\delta$	7·904 981 8
		δ(Eq. 2100)	89° 32′ 22″·66

It may be inferred from these results that the pole will pass the star early in 2102 at a distance of 27′ 36″·7.

Section III. Development of the Coordinates of a Star in Powers of the Time.

145. The rigorous methods developed in the two preceding sections are necessary when, owing to the great length of the interval through which the reduction extends, or to the high declination of the star, the change of its coordinates is an appreciable fraction of its polar distance. In ordinary cases the method of developing the coordinates of the star in a series proceeding according to the powers of the time is generally adopted.

Our first problem is to express the rate of change of the coordinates in terms of the elements of position and motion of the star and of the pole. Referring to the equations (16), and treating the interval of time and the motions between the two epochs as infinitesimal, we see from § 138, Eq. (16)-(19), that

$$
\begin{aligned}
a &\text{ reduces to } \alpha + \text{an infinitesimal},\\
\theta &\text{ reduces to } ndt,\\
p &\text{ becomes } ndt \tan\delta,\\
a'-a &\text{ reduces to } p\sin\alpha = ndt \sin\alpha \tan\delta,\\
\zeta_0+z &\text{ reduces to } mdt. \quad \text{(See § 125, Eq. 14.)}
\end{aligned}
$$

We therefore have

$$\alpha'-\alpha=(m+n\sin\alpha\tan\delta)dt.$$

Also, since $a'-a$ becomes infinitesimal,

$$\tfrac{1}{2}(a'+a) \text{ becomes } \alpha + \text{an infinitesimal},$$

so that

$$\delta'-\delta=ndt\cos\alpha.$$

Adding the proper motions, the differential coefficients of the coordinates as to the time become

$$\left.\begin{aligned}D_t\alpha &= m+n\sin\alpha\tan\delta+\mu_\alpha \equiv p_\alpha+\mu_\alpha\\ D_t\delta &= n\cos\alpha+\mu_\delta \equiv p_\delta+\mu_\delta\end{aligned}\right\}, \quad \text{............(28)}$$

which will be the coefficients of t in the development.

146. The secular variations.

To form the coefficients of the second power of the time, we have to differentiate these last expressions as to the time. Taking first the precessions p_α and p_δ, in (28), we find

$$\left.\begin{aligned}D_t p_\alpha &= D_t m+\sin\alpha\tan\delta\, D_t n+n(p_\alpha+\mu_\alpha)\cos\alpha\tan\delta\\ &\qquad +n(p_\delta+\mu_\delta)\sin\alpha\sec^2\delta\\ D_t p_\delta &= \cos\alpha\, D_t n-(p_\alpha+\mu_\alpha)n\sin\alpha\end{aligned}\right\} \text{....(29)}$$

The corresponding changes in μ_α and μ_δ comprise two parts: one due to the proper motion of the star, found in § 137, the other to precession. The combined effect of the two motions upon the proper motion itself may be found by the equations (8) and (27), taking S in the latter as infinitesimal. We then have

$$\sin S = S = ndt\sin\alpha\sec\delta, \quad \text{....................(30)}$$

$$\cos S = 1,$$

$$\mu_\alpha'\cos\delta'-\mu_\alpha\cos\delta=\mu_\delta S,$$

or, since the first member of this equation is the infinitesimal increment of $\mu_\alpha \cos\delta$,

$$\cos\delta\, d\mu_\alpha - \mu_\alpha \sin\delta\, d\delta = \mu_\delta n\, dt \sin\alpha \sec\delta.$$

In $d\delta$ we are to include only the precession p_δ. Hence

$$\cos\delta\, D_t\mu_\alpha = \mu_\alpha n \cos\alpha \sin\delta + \mu_\delta n \sin\alpha \sec\delta. \quad \text{............(31)}$$

Dividing by $\cos\delta$, we have the required variation.

We have, in like manner, from (27) and (30), for the infinitesimal increment of μ_δ,

$$\mu_\delta' - \mu_\delta = d\mu_\delta = -\mu_\alpha n\, dt \sin\alpha,$$

whence

$$D_t\mu_\delta = -\mu_\alpha n \sin\alpha. \quad \text{..........................(32)}$$

The variations (31) and (32) are those due to the precession alone. Adding them to the corresponding values (8), which give the variations due to the proper motions alone, we find for the entire variations of the proper motions,

$$\left.\begin{aligned} D_t\mu_\alpha &= (\mu_\alpha n \cos\alpha + 2\mu_\alpha\mu_\delta)\tan\delta \\ &\quad + \mu_\delta n \sin\alpha \sec^2\delta \\ D_t\mu_\delta &= -\mu_\alpha n \sin\alpha - \mu_\alpha^2 \sin\delta\cos\delta \end{aligned}\right\}. \quad \text{................(33)}$$

The sum of the first equations of (29) and (33) gives the second derivative of the R.A.:

$$\left.\begin{aligned} \frac{d^2\alpha}{dt^2} &= D_t m + D_t n \sin\alpha \tan\delta \\ &\quad + n(p_\alpha + 2\mu_\alpha)\cos\alpha \tan\delta \\ &\quad + n(p_\delta + 2\mu_\delta)\sin\alpha \sec^2\delta \\ &\quad + 2\mu_\alpha\mu_\delta \tan\delta \end{aligned}\right\}. \quad \text{............(34)}$$

For the declination, we find in the same way,

$$\frac{d^2\delta}{dt^2} = D_t n \cos\alpha - n(p_\alpha + 2\mu_\alpha)\sin\alpha - \tfrac{1}{2}\mu_\alpha^2 \sin 2\delta. \quad \text{....(35)}$$

In practical application we reduce the constants which enter into these expressions to numbers. It will also be found convenient in the terms of each expression which contains $D_t n$ to make the substitution

$$\sin\alpha \tan\delta = \frac{p_\alpha - m}{n},$$

$$\cos\alpha = \frac{p_\delta}{n}.$$

These terms then become,

$$\text{in R.A.};\ -\frac{m}{n}D_t n + p_\alpha \frac{D_t n}{n};$$

$$\text{in Dec.};\ \frac{p_\delta}{n} D_t n.$$

In astronomical literature it is common to express m and n with reference to the year as the unit. But for the longer intervals over which the reductions of stars must hereafter extend, it will be more convenient to adopt the solar century as the unit. This is especially desirable in the case of the proper motions, because, in the great majority of cases, the annual motions are so small as to require an inconvenient number of 0's after the decimal in their expression. We shall therefore compute the numerical coefficients in terms of the century as the unit, and, to avoid any possible confusion, write the centennial values of m and n as

$$m_c = 100m,$$
$$n_c = 100n,$$

thus retaining m and n as the annual values.

We take 1900 as the epoch for which the coefficients are to be given. For this epoch we have, from Appendix III.,

$$m_c = 4608''\cdot50 \quad = 307\cdot234 \text{ s.},$$
$$n_c = 2004\ \cdot68 \quad = 133\cdot646,$$
$$\frac{dm_c}{dt} = \quad +2\ \cdot79 \quad = +0\cdot186,$$
$$\frac{dn_c}{dt} = \quad -0\ \cdot853 = -0\cdot057.$$

In the expressions of the required derivatives as above written, the quantities are supposed to be expressed in homogeneous units, say seconds of arc. But in astronomical practice all the terms relative to the R.A. are expressed in seconds of time. We must therefore multiply or divide the coefficients by 15 in such a way that, in the second members, p_α and μ_α shall be expressed in time and p_δ and μ_δ in arc, while the results shall give the derivatives of α in time and those of δ in arc.

Thus, assuming that p_α and μ_α are expressed in time, we first multiply them by 15 to reduce them to arc. Then, all the terms will give the values of the first members in seconds of arc.

Then, to reduce the second derivative of α to time, we divide all its terms by 15. In the last term of each expression, which is of two dimensions in μ, we must, to produce homogeneity, multiply the coefficients by $\sin 1''$. In this way the reduction to numbers gives us

$$\left.\begin{aligned}\frac{d^2\alpha}{dT^2} = 0{\cdot}317\text{ s.} - [6{\cdot}6289]p_\alpha \\ + [7{\cdot}9876](p_\alpha + 2\mu_\alpha)\cos\alpha\tan\delta \\ + [6{\cdot}8115](p_\delta + 2\mu_\delta)\sin\alpha\sec^2\delta \\ + [4{\cdot}9866]\mu_\alpha\mu_\delta\tan\delta\end{aligned}\right\}\quad \ldots\ldots\ldots\ldots(36)$$

$$\left.\begin{aligned}\frac{d^2\delta}{dT^2} = -[6{\cdot}6289]p_\delta \\ -[9{\cdot}1637](p_\alpha + 2\mu_\alpha)\sin\alpha \\ -[6{\cdot}7367]\mu_\alpha^2\sin 2\delta\end{aligned}\right\}\quad \ldots\ldots\ldots\ldots(37)$$

In these expressions all the logarithms are to be diminished by 10. It may be noted that the last term in these expressions can scarcely be sensible except in the extreme case, when $\mu\tan\delta$ amounts to several hundred seconds of arc.

If a great number of these quantities are to be computed, the work may be facilitated by tabulating the quantities:

$$\left.\begin{aligned}A &= [7{\cdot}9876]\tan\delta \\ B &= [6{\cdot}8115]\sec^2\delta \\ C_\alpha &= 0{\cdot}317 - [6{\cdot}6289]p_\alpha \\ C_\delta &= -[6{\cdot}6289]p_\delta = -[9{\cdot}9310]\cos\alpha\end{aligned}\right\}\quad \ldots\ldots\ldots\ldots(38)$$

Tables for A, B, C_α, and C_δ are found in Appendix III.

147. It is the common practice in catalogues of stars to give the annual variations, or precessions, and the secular variations of these quantities, that is, their rate of change per century. These secular variations are equal to the preceding ones divided by 100.

When the century is taken as the unit, as in the case of the preceding development, the expression for the first two terms of the reduction to an epoch T is:

$$\alpha - \alpha_0 = T\frac{d\alpha}{dt} + \tfrac{1}{2}T^2\frac{d^2\alpha}{dt^2}$$
$$= T\left\{\frac{d\alpha}{dt} + \tfrac{1}{2}T\frac{d^2\alpha}{dt^2}\right\}.$$

When annual variations and secular variations are given, the form is:

$$\alpha - \alpha_0 = t\left(\frac{d\alpha}{dt} + \frac{1}{2}\,\frac{t}{100}\,\text{sec. var.}\right).$$

In either case the result may be obtained by multiplying the variation for the mid-epoch by the elapsed time.

148. The third term of the reduction.

To obtain the coefficients of t^3 in the expression of the coordinates, we have to differentiate the expression for the second derivative. If the effect of proper motion is included, the formulae thus derived are too prolix for practical use.

It may be doubted whether, in ordinary practice, the actual computation of this term is the best course to follow. The writer has always found the easiest course to be to compute a second value of the second term for the terminal epoch, using an approximate position of the star, and then making use of the following simple form, easily derived by taking the half sum of the two Taylor's series formed by interchanging the epochs, namely

$$\alpha_1 - \alpha_0 = t\frac{d\alpha_0}{dt} + \frac{t^2}{2}\frac{d^2\alpha_0}{dt^2} + \frac{t^3}{6}\frac{d^3\alpha}{dt^3}$$
$$= t\frac{d\alpha_1}{dt} - \frac{t^2}{2}\frac{d^2\alpha_1}{dt^2} + \frac{t^3}{6}\frac{d^3\alpha}{dt^3},$$

and substituting

$$\frac{d^2\alpha_1}{dt^2} - \frac{d^2\alpha_0}{dt^2} = t\frac{d^3\alpha}{dt^3},$$

$$\alpha_1 - \alpha_0 = \tfrac{1}{2}t\left[\frac{d\alpha_0}{dt} + \frac{d\alpha_1}{dt}\right] - \frac{t^2}{12}\left[\frac{d^2\alpha_1}{dt^2} - \frac{d^2\alpha_0}{dt^2}\right].$$

In the case of an isolated star, it may be yet easier to compute the precession for three equidistant epochs, those of reduction, and the mid-epoch. Calling the precessions, in the order of time, p_0, p_1 and p_2, the expression

$$\alpha_2 - \alpha_0 = \frac{t}{6}(p_0 + 4p_1 + p_2)$$

will then give the total reduction. This method was adopted by Auwers in reducing the positions of the Bradley stars from 1755 to 1865.

If, for any reason, this avoidance of the third term is either inaccurate or troublesome, we may always have recourse to the rigorous trigonometric reduction.

As an example of the computation and reductions, let us take the position and proper motion of 1830 Groombridge for 1875, as given in § 144, and develop the R.A. in powers of the time to T^2. We find, from these data, the precessional motions in Appendix III., and the formulae and tables for the secular variations, the motions for 1875 and 1910, as in the following computation:

	1875·0.	1910·0.
R.A., α =	176° 26′ 31″·8	176° 56′ 56″·2
Dec., δ =	38 36 55 ·6	38 21 52 ·8
$\sin\alpha$	8·792 784	8·726 122
$\tan\delta$	9·902 401	9·898 498
$\log n_c$	2·126 001	2·125 941
	0·821 186	0·750 561
$n_c \sin\alpha \tan\delta$	6·625 s.	5·631 s.
m_c	307·187	307·252
p_α	313·812	312·883
μ_α	34·198	34·040
$D_t\alpha$	348·010	346·923
$\cos\alpha$	9·999 162$_n$	9·999 384$_n$
$\log p_\delta$	3·301 254$_n$	3·301 412$_n$
$p_\alpha + 2\mu_\alpha$	382·21 s.	380·96 s.
p_δ	– 2001″·03	– 2001″·76
μ_δ	– 577 ·97	– 578 ·29
$D_t\delta$	– 2579 ·00	– 2580 ·05
$p_\delta + 2\mu_\delta$	– 3156 ·97	– 3158 ·34

	1875·0.	1910·0.
$\log(p_\alpha + 2\mu_\alpha)$	2·5823	2·5809
$\log \cos \alpha$	$9{\cdot}9992_n$	$9{\cdot}9994_n$
$\log A$ (Tab.)	7·8900	7·8861
$\log (2)$	$0{\cdot}4715_n$	$0{\cdot}4664_n$
$\log(p_\delta + 2\mu_\delta)$	$3{\cdot}4993_n$	$3{\cdot}4995_n$
$\log \sin \alpha$	8·7928	8·7261
$\log B$ (Tab.)	7·0258	7·0228
$\log (3)$	$9{\cdot}3179_n$	$9{\cdot}2484_n$
$\log \mu_\alpha$	1·534	1·534
$\log \mu_\delta$	2·762	2·762
$\tan \delta$	9·902	9·898
	4·198	4·194
log coeff.	4·987	
$\log (4)$	9·185	9·181
(1)	+0·183	+0·184
(2)	−2·961	−2·927
(3)	−0·208	−0·177
(4)	−0·153	−0·152
$D_t^2\alpha$	−3·139	−3·072

149. Precession in longitude and latitude.

We now investigate the instantaneous rate of change in the longitude and latitude of a star due to the precessional motion.

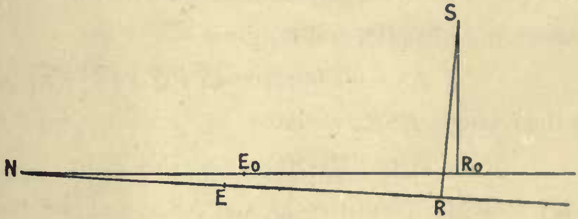

FIG. 34.

Let NR_0 be the position of the ecliptic at the initial moment; NR the position at the moment following; E_0, E the two equinoxes; SR_0, SR perpendiculars dropped from the position of the star S upon the ecliptics at the two moments;

$$\left.\begin{array}{l}\lambda_0,\ \lambda, \text{ the longitudes}\\ \beta_0,\ \beta, \text{ the latitudes}\end{array}\right\}\text{at the two moments.}$$

We then have
$$\lambda_0 = E_0R_0 = NR_0 - NE_0,$$
$$\lambda = ER = NR - NE,$$
$$\beta_0 = SR_0,$$
$$\beta = SR;$$

and for the increment of λ,

$$\Delta\lambda = NR - NR_0 + NE_0 - NE. \quad \text{.................}(a)$$

In the infinitesimal triangles SRR_0 and NRR_0, using the notation of § 122, we have

$$\kappa = \text{angle } R_0NR,$$
$$l = NE_0 - NE = \text{general precession},$$
$$N_0 = \text{arc } NE_0.$$

A study of the relations of these triangles, putting S for the infinitesimal angle at S, and applying Theorem ii. of § 7, leads to the equation
$$NR - NR_0 = -S\sin\beta, \text{.........................}(b)$$

while, noting that R and R_0 are right angles, Theorem iii. gives the equation
$$S\cos\beta = \kappa\cos NR = \kappa\cos(\lambda + N_0),$$
whence
$$S = \kappa\sec\beta\cos(\lambda + N_0). \quad \text{.......................}(c)$$

A comparison of (a), (b), and (c) gives

$$\Delta\lambda = -\kappa\tan\beta\cos(\lambda + N_0) + l. \quad \text{............}(39)$$

From the triangle RSR_0, we have

$$\Delta\beta = SR - SR_0 = \kappa\sin(\lambda + N_0). \quad \text{............}(40)$$

We may treat the quantities $\Delta\lambda$ and $\Delta\beta$, κ, and l as derivatives with respect to the time, in accordance with the methods of Chapter IX. For the epoch 1850 we have, from (4) of § 123,

$$N_0 = 6^\circ\ 30'{\cdot}32.$$

Interpolating to 1900, the epoch now most generally used, we have

	$N_0 = 6° \; 44'{\cdot}81$ (Eq. 1850)
Prec. for 50 years,	$-0 \;\; 41{\cdot}87$
N_0 for 1900,	$6° \;\; 2'{\cdot}94$

We find from the same data,

$$\kappa \text{ for } 1900 = 47''{\cdot}107.$$

The complete expression for the rate of change at 1900 due to precession alone now becomes

$$\frac{d\lambda}{dT} = -47''{\cdot}11 \tan\beta \cos(\lambda + N_0) + l \text{ for } 1900,$$

$$\frac{d\beta}{dT} = 47''{\cdot}11 \sin(\lambda + N_0), \quad \text{.................................}(41)$$

the unit of time being the solar century.

These equations give only the rate of change for 1900, or the coefficients of the first power of T. To find the cofficients of the second power of the time we must differentiate the expressions (41) as to the time.

The angle $\lambda + N_0$ is the distance from the instantaneous node of the moving on the fixed ecliptic to the projection of the star on the ecliptic. It therefore changes only in consequence of the motion of this node and the proper motion of the star. The former is, from (4) of § 123,

$$\frac{dN_0}{dT} = 29'{\cdot}0,$$

and the centennial motion of $\lambda + N_0$ is

$$\mu_\lambda + 29'{\cdot}0 = \mu_\lambda'' \sin 1'' + 0{\cdot}008\,43,$$

μ_λ being the proper motion in longitude.

We also have $$\frac{d\kappa}{dT} = -0''{\cdot}068,$$

and, by differentiation of (40) and substitution,

$$\frac{d^2\beta}{dT^2} = 0''{\cdot}40 \cos(\lambda + N_0) - 0''{\cdot}07 \sin(\lambda + N_0)$$
$$= 0''{\cdot}40 \cos(\lambda + 16°{\cdot}2),$$

the effect of proper motion being neglected.

In a similar way we find, for the part of the second derivative of λ arising from the variation of λ and N_0,

$$0''{\cdot}40 \tan \beta \cos (\lambda + 16^\circ{\cdot}2).$$

The motion of β gives rise to a term having the coefficient $0''{\cdot}005$. As there is rarely any occasion for using the ecliptic coordinates of stars outside the limits of the zodiac, this term may be dropped.

NOTES AND REFERENCES.

Two sets of tables have recently been published for the rapid computation of the annual precessions of the stars, the secular variations and the coefficients of the third power of the time. They are :—

DOWNING, A. M. W., *Precessional Tables adapted to Newcomb's value of the Precessional Constant and Reduced to the Epoch* 1910. Edinburgh, Neill & Co., 1899.

BECKER, *Tafeln zur Berechnung der Praecession* (Extract from the Annals of the Strassburg Observatory, volume ii.). Karlsruhe, G. Broun, 1898.

Becker's tables are based on the Struve-Peters values of the precessional motions for the fundamental epoch 1900. In connection with the tables is given the reduction to the new adopted values of the precessional motions. The secular variations in the two theories differ but slightly. The third term in the precession, that containing the factor T^3, is scarcely sensible for declinations less than 40°, unless the reduction extends over more than one-half a century. The writer conceives that, where account has to be taken of this term, it will be easier to compute the trigonometric reduction by the tables of the present work than to use the third term. But if it is desired to use this term tables of the coefficient of T^3 itself with the double argument R.A. and Dec. will be found in some of the Introductions to the star catalogues of the Astronomische Gesellschaft, quoted in chapter xiii.

The fundamental catalogue in *Astronomical Papers of the American Ephemeris*, vol. viii., gives tables for the trigonometric reduction to six places of decimals, but they do not extend to so high a declination as those in Appendix of the present work.

CHAPTER XI.

REDUCTION TO APPARENT PLACE.

Section I. Reduction to Terms of the First Order.

150. Reduction for nutation.

The theory of nutation, or of the revolution of the true celestial pole around the mean pole, has been developed in Chapter IX. We have now to determine the effect of this motion upon the R.A. and Dec. of a heavenly body. The relation of the true to the mean pole is expressed by the two quantities:

$\Delta\psi$, nutation in longitude.

$\Delta\epsilon$, the nutation of the obliquity of the ecliptic.

The principal terms of the original expressions from which $\Delta\psi$ and $\Delta\epsilon$ are derived have been given in Chapter IX., § 134, and are more completely tabulated in the Appendix. The fundamental data derived from them may be found in the annual Ephemerides.

Since the nutation does not affect the position of the ecliptic itself, the latitude of a heavenly body is not affected by it. For the same reason the foot of the perpendicular from the body to the ecliptic, and therefore the position of this foot, remains unchanged. Hence the effect upon the longitude of a body is only to increase it by the quantity $\Delta\psi$.

151. Nutation in R.A. and Dec.

The effect of nutation upon the R.A. and Dec. when the star is not very near the pole is so small that its powers may be neglected. It is then at once obtained from the formulae of § 54 by substituting in the equations (27) for $d\lambda$ and $d\epsilon$ the values of $\Delta\psi$ and $\Delta\epsilon$ produced by the nutation. The following are the terms of (27) which thus come into use:

$$\left.\begin{aligned}\cos\delta\Delta\alpha &= \cos S\cos\beta\Delta\lambda - \sin\delta\cos\alpha\Delta\epsilon\\ \Delta\delta &= \sin S\cos\beta\Delta\lambda + \sin\alpha\Delta\epsilon\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(1)$$

From the parts of the triangle *EPS* in § 51, we have

$$\left.\begin{aligned}\sin S\cos\beta &= \cos\alpha\sin\epsilon\\ \cos S\cos\beta &= \cos\epsilon\cos\delta + \sin\epsilon\sin\delta\sin\alpha\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(2)$$

Substituting (2) in (1) and putting $\Delta\psi$ for $\Delta\lambda$, we find, for the nutation of the R.A. and Dec.,

$$\left.\begin{aligned}\Delta\alpha &= (\cos\epsilon + \sin\epsilon\sin\alpha\tan\delta)\Delta\psi - \cos\alpha\tan\delta\Delta\epsilon\\ \Delta\delta &= \cos\alpha\sin\epsilon\Delta\psi + \sin\alpha\Delta\epsilon\end{aligned}\right\}\ \ldots\ldots(3)$$

In practice the reduction for nutation is, in the case of the fixed stars, combined with the effect of precession from the beginning of the solar year. As already mentioned, it is the universal astronomical practice to refer the mean places of the fixed stars to the equinox and equator of the beginning of some such year. Then, instead of dividing the reduction for precession and nutation into the two parts,

Precession to date + nutation,

they are divided into

Precession to beginning of solar year
+ (Precession from beginning of year to date + nutation).

The two reductions in parentheses are combined into one in the following way:

Putting τ for the elapsed fraction of the solar year, the changes in the coordinates of the star due to precession from the beginning of the year through the time τ are, neglecting the secular variation,

$$\left.\begin{aligned}\Delta\alpha &= (m + n\sin\alpha\tan\delta)\tau\\ \Delta\delta &= n\cos\alpha\,.\,\tau\end{aligned}\right\},\ \ldots\ldots\ldots\ldots(4)$$

where m and n have the following values (§ 125)

$$\left.\begin{aligned}m &= p\cos\epsilon - \lambda'\\ n &= p\sin\epsilon\end{aligned}\right\},\ \ldots\ldots\ldots\ldots(5)$$

p being the annual rate of luni-solar precession.

The corrections (4) with the substitution of (5) are now to be combined with (3), the nutation. Putting, for the moment, F for the coefficient of $\Delta\psi$ in the first equation (3),

$$F = \cos\epsilon + \sin\epsilon \sin\alpha \tan\delta.$$

It will be seen from (4) and (5) that

$$m + n\sin\alpha\tan\delta = pF - \lambda',$$

whence
$$F = \frac{m + n\sin\alpha\tan\delta}{p} + \frac{\lambda'}{p}.$$

The sum of the terms of $\Delta\alpha$ in (4) and the first equation of (3) gives for the total change in R.A., due to the combined effect of nutation and precession from the beginning of the year,

$$\Delta\alpha = \left(\tau + \frac{\Delta\psi}{p}\right)(m + n\sin\alpha\tan\delta) + \lambda'\frac{\Delta\psi}{p}. \quad \text{.................(6)}$$

So, if we put
$$\left.\begin{aligned} A &= \tau + \frac{\Delta\psi}{p} \\ a &= m + n\sin\alpha\tan\delta \\ E &= \lambda'\frac{\Delta\psi}{p} \end{aligned}\right\}, \quad \text{.........................(7)}$$

we shall have the effects of precession from the beginning of the year and nutation in longitude combined in the simple expression
$$\Delta\alpha = Aa + E. \quad \text{.............................(8)}$$

For the declination the values of $\Delta\delta$ in (3) and (4) may be combined in a similar way. We have from (5),

$$\sin\epsilon\cos\alpha = \frac{n}{p}\cos\alpha,$$

and thus the sum of the two terms in question may be written

$$\Delta\delta = \left(\tau + \frac{\Delta\psi}{p}\right)n\cos\alpha. \quad \text{.........................(9)}$$

So, if we put
$$a' = n\cos\alpha, \quad \text{..............................(10)}$$

we shall have the effects of precession from the beginning of the year and nutation in declination combined in the simple form
$$\Delta\delta = Aa', \quad \text{..............................(11)}$$

The effect of nutation in obliquity may be expressed in the same way. The practice is to put

$$\left.\begin{aligned} B &= -\Delta\epsilon \\ b &= \cos\alpha \tan\delta \text{ (or numerically } \tfrac{1}{15}\cos\alpha\tan\delta) \\ b' &= -\sin\alpha \end{aligned}\right\} \quad\text{.......(12)}$$

The coefficient b is divided by 15 in order that $\Delta\alpha$ may be expressed in time.

We then have, for the nutation in R.A. and Dec. depending on $\Delta\epsilon$,

$$\left.\begin{aligned} \Delta\alpha &= Bb \\ \Delta\delta &= Bb' \end{aligned}\right\} \quad\text{..............................(13)}$$

152. Reduction for aberration.

The formulae for the effect of aberration upon the coordinates of a fixed star, considered as infinitesimal, are found in Chapter VII., Eq. (13). We note that the terms of $\cos\delta\Delta\alpha$ containing e as a factor are functions of α, π, and ϵ, and being nearly constant in the case of any one star are regarded as included in the mean R.A. of the star, and left out of consideration. We thus have for the aberration:

$$\left.\begin{aligned} \Delta\alpha &= -\kappa\sin\odot\sin\alpha\sec\delta - \kappa\cos\odot\cos\epsilon\cos\alpha\sec\delta \\ \Delta\delta &= -\kappa\cos\odot(\sin\epsilon\cos\delta - \cos\epsilon\sin\alpha\sin\delta) \\ &\quad -\kappa\sin\odot\cos\alpha\sin\delta \end{aligned}\right\} \quad\text{...(14)}$$

If we put

$$\left.\begin{aligned} C &= -\kappa\cos\epsilon\cos\odot \\ D &= -\kappa\sin\odot \\ c &= \cos\alpha\sec\delta \div 15 \text{ (to reduce to time)} \\ d &= \sin\alpha\sec\delta \div 15 \text{ (to reduce to time)} \\ c' &= \tan\epsilon\cos\delta - \sin\alpha\sin\delta \\ d' &= \cos\alpha\sin\delta \end{aligned}\right\}, \quad\text{.........(15)}$$

these equations become

$$\left.\begin{aligned} \Delta\alpha &= Cc + Dd \\ \Delta\delta &= Cc' + Dd' \end{aligned}\right\}, \quad\text{.........................(16)}$$

which is the simplest form of expressing the aberration when its powers are dropped.

153. Reduction for parallax.

When we take into account the effect of the annual parallax of a star upon its R.A. and Dec., we must conceive its mean place to be referred to the sun, and then find the reduction to the earth. If r be the distance of the star from the sun, and X, Y, Z the rectangular equatorial coordinates of the sun; and if we designate the geocentric coordinates of the star by accents, they will be given by the equations

$$\left.\begin{aligned} x' &= r'\cos\delta'\cos\alpha' = r\cos\delta\cos\alpha + X \\ y' &= r'\cos\delta'\sin\alpha' = r\cos\delta\sin\alpha + Y \\ z' &= r'\sin\delta' = r\sin\delta + Z \end{aligned}\right\}. \quad \text{.........(17)}$$

Owing to the vast distance of the stars and the consequent great value of r, we may treat X, Y, and Z as infinitesimal increments of x', y', and z' respectively, and determine the corresponding increments of α and δ by the equations (4) of § 48, putting α and δ for λ and β, and X, Y, Z for dx, dy, and dz respectively. We also put π, the annual parallax of the star, that is, the angle subtended by the earth's mean distance from the sun when seen from the star, which makes

$$r\sin\pi = 1.$$

We thus derive, from the equations (4a) of § 48,

$$\left.\begin{aligned} \cos\delta\,\Delta\alpha &= \sin\pi(-X\sin\alpha + Y\cos\alpha) \\ \Delta\delta &= \sin\pi(Z\cos\delta - X\sin\delta\cos\alpha - Y\sin\delta\sin\alpha) \end{aligned}\right\}. \quad \text{...(18)}$$

These expressions may be reduced to the form of the other star corrections in the following way. Putting, as before, $\odot$ for the sun's true longitude and R for its radius vector, we have

$$\begin{aligned} X &= R\cos\odot, \\ Y &= R\cos\epsilon\sin\odot, \\ Z &= R\sin\epsilon\sin\odot. \end{aligned}$$

Substituting these values in (18) and putting π for its sine, we find

$$\left.\begin{aligned} \Delta\alpha &= R\pi(-\cos\odot\sin\alpha + \cos\epsilon\sin\odot\cos\alpha)\sec\delta \\ \Delta\delta &= R\pi(-\cos\odot\sin\delta\cos\alpha - \cos\epsilon\sin\odot\sin\delta\sin\alpha \\ &\qquad + \sin\epsilon\sin\odot\cos\delta) \end{aligned}\right\}. \quad (19)$$

These can be expressed by means of the same star constants as are used in computing the aberration, after multiplying them by the parallax. That is, if we put, as functions of the coordinates of the star and of its parallax, using π'' as the parallax in seconds of arc and $\pi_s = \pi'' \div 15 = \pi$ in seconds of time,

$$\left.\begin{aligned} c_1 &= \pi_s \cos\alpha \sec\delta = \pi'' c \\ d_1 &= \pi_s \sin\alpha \sec\delta = \pi'' d \\ c'_1 &= \pi'' (\tan\epsilon \cos\delta - \sin\alpha \sin\delta) = \pi'' c' \\ d'_1 &= \pi'' \cos\alpha \sin\delta = \pi'' d' \end{aligned}\right\}, \quad \text{.........(20)}$$

and, as factors depending on the sun's longitude,

$$\left.\begin{aligned} C_1 &= R \cos\epsilon \sin \odot \\ D_1 &= -R \cos \odot \end{aligned}\right\}, \quad \text{......................(21)}$$

we shall have

$$\left.\begin{aligned} \Delta\alpha &= C_1 c_1 + D_1 d_1 \\ \Delta\delta &= C_1 c'_1 + D_1 d'_1 \end{aligned}\right\}. \quad \text{......................(22)}$$

154. Combination of the reductions.

We next show how the preceding reductions may best be combined. Omitting the reduction for parallax, which need be taken account of only in a few exceptional cases, the reduction of a star from its mean place at the beginning of a year to its apparent place at any time during the year may be computed by the formulae (8), (11), (13), and (16). Adding the correction for proper motion from the beginning of the year to the date, we shall have

$$\left.\begin{aligned} \Delta\alpha &= Aa + Bb + Cc + Dd + E + \mu_\alpha \tau \\ \Delta\delta &= Aa' + Bb' + Cc' + Dd' + \mu_\delta \tau \end{aligned}\right\}. \quad \text{...............(23)}$$

The coefficients A, B, C, D, and E are functions of the time but independent of the position of the star. Hence, on any one date, they are the same for all the stars. They are known in astronomy as the *Besselian day numbers*, after the great Bessel, who first introduced them into use. Their values for every day of the year are found in the annual ephemeris.

On the other hand, the numbers a, a', b, etc., being functions of the place of the star, are regarded as constants for greater or less periods of time. The logarithms of these constants for individual stars are given in some of the catalogues, so as to save the astronomer using the catalogue the trouble of computing them. But as the position of every star varies from year to year, it is a question how long any such constants can be used without important error. The general rule is that, in the case of stars near the equator, say those whose declination is less than 45°, the constants may be used for several years unchanged. But as we approach the pole, the period during which no change need be made becomes shorter and shorter.

Some of the catalogues give in addition to the constants for a given epoch either their values at some other epoch or the annual change in the last figure of the logarithm. With such catalogues reductions can be made without danger of error.

155. Independent day numbers.

There is another form of reduction to apparent place which is much used when sufficiently accurate values of the star constants are not at hand. In the equations (8), (11), and (13) let us substitute for a, a', b, and b' their values as given in (7), (10), and (12). The reduction for precession and nutation thus becomes

$$\left.\begin{aligned}\Delta\alpha &= Am+(An\sin\alpha+B\cos\alpha)\tan\delta+E\\ \Delta\delta &= An\cos\alpha-B\sin\alpha\end{aligned}\right\}. \qquad \text{.........(24)}$$

In the same way, the terms of aberration as found in (14) and (16) may be written

$$\left.\begin{aligned}\Delta\alpha &= (C\cos\alpha+D\sin\alpha)\sec\delta\\ \Delta\delta &= C\tan\epsilon\cos\delta+(D\cos\alpha-C\sin\alpha)\sin\delta\end{aligned}\right\}. \qquad \text{.........(25)}$$

In the second term of (24) let us replace A and B by the quantities g and G, determined by the equations

$$\left.\begin{aligned}g\sin G &= B\\ g\cos G &= An\end{aligned}\right\}; \qquad \text{.........................(26)}$$

we shall then have

$$An\sin\alpha+B\cos\alpha=g\sin(G+\alpha),$$
$$An\cos\alpha-B\sin\alpha=g\cos(G+\alpha),$$

and (24) becomes

$$\Delta\alpha=g\sin(G+\alpha)\tan\delta+Am+E,$$
$$\Delta\delta=g\cos(G+\alpha).$$

Let us also transform (25) in a similar way, determining h and H by the conditions

$$\left.\begin{aligned} h\sin H &= C \\ h\cos H &= D \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(27)$$

We then have

$$C\cos\alpha + D\sin\alpha = h\sin(H+\alpha),$$
$$D\cos\alpha - C\sin\alpha = h\cos(H+\alpha),$$

and (26) becomes

$$\Delta\alpha = h\sin(H+\alpha)\sec\delta,$$
$$\Delta\delta = h\cos(H+\alpha)\sin\delta + C\tan\epsilon\cos\delta.$$

Let us also put

$$\left.\begin{aligned} f &= Am + E \\ i &= C\tan\epsilon \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28)$$

By these substitutions the total reductions for nutation and aberration, adding in the proper motion, become

$$\left.\begin{aligned} \Delta\alpha &= f + g\sin(G+\alpha)\tan\delta + h\sin(H+\alpha)\sec\delta + \mu_\alpha\tau \\ \Delta\delta &= g\cos(G+\alpha) + h\cos(H+\alpha)\sin\delta + i\cos\delta + \mu_\delta\tau \end{aligned}\right\}, \ldots(29)$$

which may be used instead of (23). The numbers f, g, etc., known as *independent day numbers*, are given in the Ephemerides.

The choice between the use of Besselian and of the independent day numbers depends upon the special character of the work. The general rule is that, if the problem is to compute a number of positions of the same star, say an ephemeris for an entire year, the Besselian numbers will be the most convenient. This advantage will hold true even for a single apparent place, if the star constants a, b, etc., are already at hand. But if these constants have to be computed, and especially if the problem is to reduce a large number of stars to apparent place at the same date, the independent day numbers will give the most rapid computation.*

* The computer using the *British Nautical Almanac* or the *Connaissance des Temps* should have in mind that the day numbers in these two publications have a different notation from that above used, which is the original one of Bessel. When these numbers were introduced into England by Baily, those expressing aberration were changed to A and B, and those for nutation to C and D. This system was also adopted in Paris. In the early years of the *American Ephemeris*

Section II. Rigorous Reduction for Close Polar Stars.

156. In the preceding method of reduction, the changes produced by precession during the fraction of the year, by nutation and by aberration, have all been treated as infinitesimals. It has therefore been assumed to be indifferent whether the mean or the apparent place of the star is used in the formulae, and quantities of higher dimensions than the first in the three changes have been dropped as unimportant. This deviation from rigour will lead to no appreciable error when the amount of the reduction is not an important fraction of the star's distance from the pole. But, however small the changes may be in themselves, there is always a certain distance from the pole within which a more rigorous process is necessary. The choice among the various methods of reduction that may be adopted in this case depends largely on the nature of the problem in hand and the degree of precision required.

The more precise methods which may be adopted are of two classes. In one a formally rigorous reduction is carried through by trigonometric methods. In the other class the reductions are developed to quantities of the second order with respect to their values. It must be noted in this connection that any method of development in powers of the reduction will fail in the immediate region of the pole, though it may be applicable to all the standard stars now in use.

In order to appreciate the degree of precision required, the fact must be borne in mind that, on account of the convergence of the meridians, as explained in § 44, the actual error in the position of a star arising from a given error of its R.A. diminishes without limit as the pole is approached. It follows that if we have in the R.A. an expression of the form

$$\Delta\alpha = k \sec\delta \text{ or } \Delta\alpha = k \tan\delta,$$

the English system was adopted. But in the *Berliner Astronomisches Jahrbuch*, and in the *American Ephemeris* after the first few years, the original notation has been used throughout, as defined in the present chapter. It may also be said that in catalogues in which polar distance is used instead of declination, especially in the British Association catalogue, the accented star constants for the declination have their sign changed in order to give the reduction of the polar distance.

then although, as the pole is approached, $\Delta\alpha$ increases without limit, the amount of correction to the actual position of the star will be measured by k only. Since it is impossible in practical measurement to gain greatly in accuracy by being near the pole, it follows that the importance of the term $k \sec \delta$ must depend on the value of k alone.

This does not apply to a correction $\Delta\delta$ in declination. If this contains a factor $\sec \delta$ or $\tan \delta$, it will increase proportionally to that function. Moreover when a term of the R.A. contains $\sec^2\delta$ or $\tan^2\delta$, the effect of the term on the position of the star increases indefinitely as the pole is approached.

157. Trigonometric reduction for nutation.

Let P be the mean pole, P' the actual pole as affected by nutation, and S the position of the star. It is indifferent whether

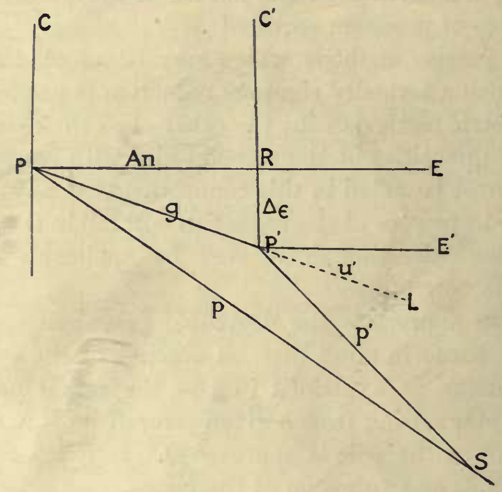

FIG. 35.

we take for P the mean pole of the date or that at the beginning of the year. It will be generally more convenient to take the pole for the beginning of the year. Then, as heretofore, the luni-solar precession to date will be combined with the term $\Delta\psi$ of the nutation.

Let CP and $C'P'$ be small arcs of the colures through P and P' and PE and $P'E'$ arcs of the circles passing through the mean and apparent equinoxes respectively. We shall then have

$$\text{Angle } CPE = \text{Angle } C'P'E' = 90°,$$
$$RP' = \Delta\epsilon.$$

In Fig. 35 the day-numbers g and G are geometrically represented, as are also the mean and reduced coordinates of the star, as follows:

$-G = RPP'$.
$-G' = E'P'L$, $P'L$ being the continuation of PP'.
$g = PP'$.
$\alpha_0 = EPS$, the mean R.A.
α', the R.A. affected by precession to date and nutation $= E'P'S$.
$\delta_0 = 90° - PS$, the mean declination.
$\delta' = 90° - P'S$, the declination affected by precession to date and nutation.

From Theorem (ii.) of differential spherical astronomy, we have, assuming that P is the pole for the beginning of the year, and using the day numbers A and B,

$$PR = (p\tau + \Delta\psi)\sin\epsilon = Ap\sin\epsilon = An.$$

In determining g and G from $p + \Delta\psi$ and $\Delta\epsilon$, we may treat the triangle RPP' as infinitesimal, because the effect of the resulting errors will be only an error of the second order in the position of the pole P', which is independent of the position of the star, and therefore does not increase when the latter is near the pole.

The angle G and the side $PP' = g$ may therefore be found from the equations

$$\left.\begin{aligned} g\sin G &= -\Delta\epsilon = B \\ g\cos G &= Ap\sin\epsilon = An \end{aligned}\right\} \quad \text{(29a)}$$

From Theorem (iii.), § 7, we have

$$C'P'L = CPP' + Ap\cos\epsilon = CPP' + f,$$

the term E in f being dropped because unimportant in this case.

Subducting equal right angles, we shall have left

$$G' = G - f.$$

In the triangle SPP' we have

$$\text{Angle } P = \alpha_0 + G,$$

$$\text{Exterior Angle } SP'L = \alpha' + G'.$$

The relations between the five parts of this triangle which have been defined give the equations

$$\sin(90° - \delta')\sin(\alpha' + G') = \sin(90° - \delta_0)\sin(\alpha_0 + G),$$

$$\sin(90° - \delta')\cos(\alpha' + G') = \cos g \sin(90° - \delta_0)\cos(\alpha_0 + G) - \sin g \cos(90° - \delta_0),$$

$$\cos(90° - \delta') = \sin g \sin(90° - \delta_0)\cos(\alpha_0 + G) + \cos g \cos(90° - \delta_0).$$

Putting, for brevity, $\quad \mathrm{a} \equiv \alpha_0 + G,$

$$\mathrm{a}' = \alpha' + G',$$

the relations become

$$\cos\delta' \sin \mathrm{a}' = \cos\delta_0 \sin \mathrm{a}, \qquad (30)$$

$$\left.\begin{aligned}\cos\delta' \cos \mathrm{a}' &= \cos g \cos\delta_0 \cos \mathrm{a} - \sin g \sin\delta_0 \\ \sin\delta' &= \sin g \cos\delta_0 \cos \mathrm{a} + \cos g \sin\delta_0\end{aligned}\right\} \qquad (30a)$$

These equations become identical in form with (14), § 138, when we write g for θ, G for ζ_0 and G' for $-z$; and may therefore be solved in the same way. But g is so minute, its maximum value being about 30″, that we may drop its powers, when not multiplied by a factor which becomes infinite at the pole, and put $\sin g = g$, $\cos g = 1$. With this change, the formulae for solving the preceding equations for α' and δ' are as follows. We accent the symbol p to avoid confusing it with the precession and put Δ_n for the increment due to nutation and precession:

$$\left.\begin{aligned}\mathrm{a} &= \alpha_0 + G. \\ p' &= g \tan\delta_0 \\ \tan\Delta_n \mathrm{a} &= \frac{p' \sin \mathrm{a}}{1 - p' \cos \mathrm{a}} \\ A_n\alpha &= \Delta_n \mathrm{a} + Ap \cos\epsilon \\ A_n\delta &= g \cos(\mathrm{a} + \tfrac{1}{2}\Delta_n \mathrm{a}) \sec \tfrac{1}{2}\Delta \mathrm{a}\end{aligned}\right\} \qquad (31)$$

By expressing g and p' in seconds of arc, computing

$$g_s = g'' \div 15,$$
$$p_s' = g_s \tan\delta,$$

we may use the Tables of Appendix IV. in the solution.

It is also to be noted that in the case of a star only a few minutes, say 5′ or less, from the pole, the rigorous equation may be necessary in the computation of δ.

158. Trigonometric reduction for aberration.

The reduction for aberration may also be expressed in the trigonometric form. We have found (§ 87) that the changes in the equatorial rectangular coordinates X_1, Y_1, Z_1 of a star produced by aberration are:

$$\left.\begin{aligned} \Delta X_1 &= R\kappa \sin \odot &&= -RD \\ \Delta Y_1 &= -R\kappa \cos\odot \cos\epsilon &&= RC \\ \Delta Z_1 &= -R\kappa \cos\odot \sin\epsilon &&= RC\tan\epsilon \end{aligned}\right\}, \qquad (32)$$

R being the distance of the star and C and D the day numbers. Expressing the spherical coordinates in terms of the rectangular ones, putting R' for the apparent distance, and

$$f = \frac{R'}{R},$$

we find that the apparent R.A. and Dec. α and δ may be derived from α' and δ' by solving the equations

$$\left.\begin{aligned} f\cos\delta\cos\alpha &= \cos\delta'\cos\alpha' - D \\ f\cos\delta\sin\alpha &= \cos\delta'\sin\alpha' + C \\ f\sin\delta &= \sin\delta' + C\tan\epsilon \end{aligned}\right\}. \qquad (33)$$

These equations may be solved like those for parallax. By cross-multiplication of the first two by $\sin\alpha'$ and $\cos\alpha'$, we find

$$\left.\begin{aligned} f\cos\delta\sin\Delta_a\alpha &= C\cos\alpha' + D\sin\alpha' = h\sin(H+\alpha') \\ f\cos\delta\cos\Delta_a\alpha &= \cos\delta' + C\sin\alpha' - D\cos\alpha' \\ &= \cos\delta' - h\cos(H+\alpha') \end{aligned}\right\}, \qquad (34)$$

where we put, for the aberration in R.A.,

$$\Delta_a\alpha = \alpha - \alpha'.$$

Forming the quotient of these equations:

$$\tan\Delta_a\alpha = \frac{h\sin(H+\alpha')\sec\delta'}{1 - h\cos(H+\alpha')\sec\delta'} \qquad (35)$$

For the declinations we add the products of the equations (34) by $\sin\frac{1}{2}\Delta_a\alpha$ and $\cos\frac{1}{2}\Delta_a\alpha$ respectively, thus obtaining

$$\left.\begin{aligned} f\cos\delta\cos\tfrac{1}{2}\Delta_a\alpha &= \cos\delta'\cos\tfrac{1}{2}\Delta_a\alpha - h\cos(H+\alpha'+\tfrac{1}{2}\Delta_a\alpha) \\ f\cos\delta &= \cos\delta' - h\cos(H+\alpha'+\tfrac{1}{2}\Delta_a\alpha)\sec\tfrac{1}{2}\Delta_a\alpha \end{aligned}\right\}. \quad (36)$$

Then, by cross-multiplication of this equation and $(33)_3$ by $\sin\delta$ and $\cos\delta$, and putting

$$\Delta_a\delta = \delta - \delta',$$

we have

$$f\sin\Delta_a\delta = C\tan\epsilon\cos\delta' + h\sin\delta'\cos(H+\alpha'+\tfrac{1}{2}\Delta_a\alpha)\sec\tfrac{1}{2}\Delta_a\alpha,$$
$$f\cos\Delta_a\delta = 1 + C\tan\epsilon\sin\delta' - h\cos\delta'\cos(H+\alpha'+\tfrac{1}{2}\Delta_a\alpha)\sec\tfrac{1}{2}\Delta_a\alpha.$$

If we compute j and J from

$$\left.\begin{aligned} j\sin J &= C\tan\epsilon \\ j\cos J &= h\cos(H+\alpha'+\tfrac{1}{2}\Delta_a\alpha)\sec\tfrac{1}{2}\Delta_a\alpha \end{aligned}\right\}, \quad \text{..........(37)}$$

the quotient of these equations will give

$$\tan\Delta_a\delta = \frac{j\sin(J+\delta')}{1-j\cos(J+\delta')}. \quad \text{................(38)}$$

The equations (31), (35), and (38) give the reduction for nutation and aberration respectively. It is to be noted that in (31) the α and δ with which we start are the mean coordinates, while, in (35) and (38), they are the coordinates affected by nutation. We may, without any drawback, reverse the order of the two corrections, computing the aberration with the mean place of the star, and then the nutation with the place as affected by aberration. As a check upon the accuracy of the work it may be well to make the computation in both these orders.

Section III. Practical Methods of Reduction.

159. Although the preceding exposition of the methods of reduction is complete so far as the theory of the work is concerned, it it necessary to minimize the labour of applying the theory by making the best use of the data in the ephemeris, and omitting all processes which are not necessary to the special problem in hand. The astronomical ephemerides give not only the day numbers for each day in the year, but ephemerides of the apparent places of several hundred fundamental stars, which will, in all ordinary cases, relieve the astronomer from the necessity of making any computations relating to the apparent places of these particular stars. But when an unusual degree of theoretical precision is required in the results, there are certain points which require attention even in using the ephemeris for this purpose. There is, in fact, when labour-saving devices are applied, a practical difficulty arising from the periods and values of the terms of nutation. These terms are, in § 134, divided into three classes according to the length of their period. In the case of the larger terms, the period is that of the moon's node, 18·6 y., or its half. Next in the order, both of length of period and of magnitude, are the annual or semi-annual terms.

Neither of these classes of terms offers any difficulty growing out of the length of period. The difficulty arises in dealing with the small terms of the third class, the length of whose periods is about a month or some fraction of a month. The largest of these is within the limit of error of all but the most refined observations, but not far enough within to be always neglected as unimportant. The method of dealing with them will be seen by a survey of the practical conditions and data of the problem.

160. Treatment of the small terms of nutation.

The astronomical ephemeris gives the apparent positions of the principal fixed stars to 0·01 s. in R.A. and 0″·1 in Dec. for every tenth day of the year. In the case of the close polar stars the positions are given for every day.

In the ten-day ephemeris it would be useless to include the terms of short period, because an interpolation of such terms to intermediate dates could not be made with accuracy. We readily see that, where the period of the term is 14 days, the term might be negative at two consecutive ten-day epochs, and pass through its maximum positive value during the interval. It follows that when the astronomer makes use of the ten-day ephemeris he must ignore these short-period terms altogether, or spend much labour in applying them. Moreover, when, as is the custom, they are included in the positions of the polar stars, but omitted from those of other stars, there is a non-homogeneity in the results which may be productive of confusion.

We begin a more special study of the conditions by noting that the terms of nutation in R.A., which are larger than those in Dec., may be divided into two classes: those which vary with the declination, having $\tan\delta$ as a factor, and those which, at any moment, are independent of the declination, and therefore the same for all declinations.

If no coordinates but equatorial ones were ever used in astronomy, the latter terms, whatever their magnitude, could be dropped out as unnecessary. We should then be referring all R.A.'s, not to the apparent equinox of the date, but to a quasi mean equinox affected by all the other inequalities, as an origin. The reason why this equinox is not adopted as the origin of R.A. is that the motions of the planets are in the first place necessarily referred to the ecliptic as the fundamental plane; and, in order to obtain a correct reduction to the equator, the actual equinox at each day, with all its inequalities, must be made use of. It is quite possible that if, following this practice so far as the original computations were concerned, the system were universally adopted of dropping constant terms of nutation from the R.A. of all heavenly bodies, using them only in the original computations where longitudes entered, it would be a simplification of our present system, which would carry with it no serious drawbacks.

No such scheme is, at present, practicable in its entirety. But at a conference held in Paris in 1896, at which the Directors of the principal astronomical ephemerides devised a uniform system of dealing with star-reductions, it was agreed to drop from the R.A. of all stars those minute constant terms of short period which are common to all the stars. A step is thus taken toward the simplification which has been suggested in the origin of Right Ascension.

Although we thus get rid of those parts of those nutation terms of short period which are common to all the stars, we do not thereby avoid the terms which vary with the declination. The celestial pole does actually go through two revolutions per month in a very small curve 0″·17 in diameter, approximating to a circle; and our instruments, being carried upon the moving earth, are affected by this motion, which must therefore be taken account of in the most refined reductions. A small correction depending on the tangent of the declination is therefore included in the ephemerides of the polar stars. This gives rise to a non-homogeneity between the star positions given for every ten days and those given for every day.

The terms in question are so minute that the practical astronomer has, in all ordinary cases, no occasion to trouble himself with them. He can use the numbers of the ephemeris with entire confidence that he will be led into no appreciable error by the lack of homogeneity. If engaged in any special research in which so small a correction is important, the ephemeris supplies all data necessary for his purpose.

161. Development of the reduction to terms of the second order.

Although the computation of the reduction by the preceding rigorous formulae will probably be found simpler than the use of a development in series, when only a single reduction is wanted, there are some purposes in which a development of the reduction is required. Unless the star to be reduced is within 5′ of either pole, a development to terms of the second order will be sufficient. When we drop terms of the third and higher orders in the development, a number of simplifications may be made in the process by dropping out all terms which, in the final result, will rise only to the third order. The following are some of the cases in which this or other simplifications may be made:

1. Since the tangent of a small arc differs from the arc itself only by a quantity of the third order, it follows that, in developing to terms of the second order, we may substitute the reduction itself for its tangent.

2. For the same reason the cosine and secant of a quantity of the order of magnitude of the reduction may be supposed equal

to unity, and therefore dropped as a factor when multiplied by the reduction itself.

3. In forming the several increments of the reductions of the first order, in order to obtain the terms of the second order, it will be sufficient to carry these terms to the first order only.

4. So far as the terms of the second order are a function of the coordinates of the star, it is indifferent whether we use the mean or apparent values of these coordinates in the expressions for such terms.

5. For the reason already mentioned, the only terms of the second order which need to be included in the R.A. are those which contain terms of two dimensions in $\sec\delta$ or $\tan\delta$. In the case of the declination all terms may be dropped which do not contain either $\tan\delta$ or $\sec\delta$ as a factor.

In forming the required increments of the second order it will be our object to first express them in terms of g, G, h, etc., and then replace these quantities by their expressions in terms of the Besselian day numbers A, B, C, D, by means of the equations (26), (27), and (28).

Following the same order as in the preceding rigorous reduction, we shall begin by forming the terms of the second order due to precession to date and nutation alone, which terms we shall designate by the symbols

$$\Delta_n^2\alpha, \quad \Delta_n^2\delta.$$

The terms of the second order due to aberration will then be found by assigning the increments $\Delta_n\alpha$, $\Delta_n\delta$ of the first order to the expressions for the reduction for aberration, and also the increments consisting of the terms of the first order in the aberration itself. The aberration-terms of the second order will then be the changes in the aberration due to these increments of the first order. The combined increments of the second order thus arising will be designated as

$$\Delta_{n,a}^2\alpha, \quad \Delta_{n,a}^2\delta, \quad \Delta_a^2\alpha, \quad \Delta_a^2\delta.$$

162. Precession and nutation.

Beginning with the terms arising from precession to date and nutation combined, we write the necessary portions of the rigorous reduction as given in (31) in the following form, where we have substituted for a its value α_0+G:

$$\left.\begin{aligned}\Delta_n\alpha&=\frac{p'\sin(G+\alpha_0)}{1-p'\cos(G+\alpha_0)}\\ \Delta_n\delta&=g\cos(G+\alpha_0+\tfrac{1}{2}\Delta_n\alpha)\end{aligned}\right\}. \qquad \text{.............(40)}$$

Neglecting $p'\cos(G+\alpha_0)$ in the denominator of the fraction, the expression for $\Delta_n\alpha$ will reduce to the reduction already found for terms of the first order. When terms of the second order only in p' are included, we may write

$$(1-p'\cos(G+\alpha_0))^{-1}=1+p'\cos(G+\alpha_0).$$

Thus the terms of the second order in the reduction of the right ascension become

$$\left.\begin{aligned}\Delta_n^2\alpha&=p'^2\sin(G+\alpha)\cos(G+\alpha)\\ &=(B\cos\alpha+An\sin\alpha)(An\cos\alpha-B\sin\alpha)\tan^2\delta\\ &=\tfrac{1}{2}\{ABn\cos 2\alpha+(A^2n^2-B^2)\sin 2\alpha\}\tan^2\delta\end{aligned}\right\}. \quad \text{...(41)}$$

For the corresponding terms in the declination we have

$$\begin{aligned}\Delta_n^2\delta&=-\tfrac{1}{2}g\sin(G+\alpha)\Delta_n\alpha\\ &=-\tfrac{1}{2}g^2\tan\delta\sin^2(G+\alpha).\end{aligned}$$

By easy reductions this becomes

$$\Delta_n^2\delta=\{-\tfrac{1}{4}g^2+\tfrac{1}{4}(A^2n^2-B^2)\cos 2\alpha-\tfrac{1}{2}ABn\sin 2\alpha\}\tan\delta. \quad (42)$$

163. Aberration.

Passing now to the aberration: in order to obtain its complete effect we have to substitute for α' and δ' in (35) and (38) the values $\alpha_0+\Delta_n\alpha$ and $\delta_0+\Delta_n\delta$. We also have to include the terms of the second order resulting immediately from the development of the denominator. The latter are, for the R.A.:

$$\begin{aligned}\Delta_a^2\alpha&=h^2\sin(H+\alpha)\cos(H+\alpha)\sec^2\delta\\ &=\{CD\cos 2\alpha+\tfrac{1}{2}(D^2-C^2)\sin 2\alpha\}\sec^2\delta.\end{aligned}$$

Here, as before, we use the symbols α and δ without farther specification, because the terms are of the second order.

For the substitution of $\Delta_n\alpha$ and $\Delta_n\delta$ we require the expressions (35) and (38) to the first order only, using

$$\Delta_a\alpha=h\sin(H+\alpha_0+\Delta_n\alpha)\sec(\delta_0+\Delta_n\delta).$$

Then

$$\Delta^2_{n,a}\alpha = h\cos(H+\alpha)\Delta_n\alpha\sec\delta + h\sin(H+\alpha)\Delta_n\delta\sec\delta\tan\delta. \quad (43)$$

We assign to $\Delta_n\alpha$ and $\Delta_n\delta$ their values (40), taken only to terms of the first order, namely

$$\Delta_n\alpha = g\sin(G+\alpha)\tan\delta = (An\sin\alpha + B\cos\alpha)\tan\delta,$$
$$\Delta_n\delta = g\cos(G+\alpha) = An\cos\alpha - B\sin\alpha.$$

These increments being substituted in (43), the latter reduces to

$$\Delta^2_{n,a}\alpha = gh\sin(G+H+2\alpha)\tan\delta\sec\delta.$$

The factor of $\tan\delta\sec\delta$ being

$$gh\sin(G+H)\cos 2\alpha + gh\cos(G+H)\sin 2\alpha,$$

we find that

$$gh\sin(G+H) = ACn + BD,$$
$$gh\cos(G+H) = ADn - BC.$$

We now obtain

$$\Delta^2_{n,a}\alpha = \{(ACn+BD)\cos 2\alpha + (ADn-BC)\sin 2\alpha\}\tan\delta\sec\delta. \quad (44)$$

Proceeding in the same way with the declination, we find that the terms of the second order in (38) are found by writing the latter in the form

$$\Delta_a\delta = j\sin(J+\delta')\{1 + j\cos(J+\delta')\},$$

and are, when aberration only is considered,

$$\Delta^2_a\delta = j^2\sin(J+\delta)\cos(J+\delta). \quad\text{.................}(45)$$

Comparing with (37), we see that neither factor of this product is increased in approaching the pole; $\Delta^2_a\delta$ may, therefore, be dropped, leaving only the nutational increment of (38) to be considered. We reduce the principal term of (38) thus:

$$\left.\begin{aligned}\Delta_a\delta &= j\sin(J+\delta')\\ &= j\sin J\cos\delta' + j\cos J\sin\delta'\end{aligned}\right\}. \quad\text{..............}(46)$$

Beginning with the aberrational increment of this expression, we see from (37) that $\sin J$ does not contain α or δ. For the increment of $\cos J$ we have

$$\Delta_a(j\cos J) = -\tfrac{1}{2}h\sin(H+\alpha)\,\Delta_a\alpha.$$

From (35), the principal value of $\Delta_a\alpha$ is

$$\Delta_a\alpha = h\sin(H+\alpha)\sec\delta.$$

We have from (27),

$$h \sin(H+\alpha) = C \cos\alpha + D \sin\alpha.$$

By these successive substitutions the increment of (46) becomes

$$\Delta_a^2\delta = -\{\tfrac{1}{4}(C^2+D^2) + \tfrac{1}{4}(C^2-D^2)\cos 2\alpha + \tfrac{1}{2}CD\sin 2\alpha\}\tan\delta. \quad (47)$$

We next assign to $\cos J$ its nutational increment through putting in (37)

$$\alpha' = \alpha_0 + \Delta_n\alpha.$$

We thus have

$$\begin{aligned}\Delta_n(j\cos J) &= -h\sin(H+\alpha)\,\Delta_n\alpha \\ &= -(C\cos\alpha + D\sin\alpha)\,\Delta_n\alpha.\end{aligned}$$

By substitution of $\Delta_n\alpha$, the increment of (46) reduces to

$$\begin{aligned}\Delta_{a,n}^2\delta = -\tfrac{1}{2}\{ADn + BC - (ADn - BC)\cos 2\alpha \\ + (ACn + BD)\sin 2\alpha\}\sin\delta\tan\delta. \quad (48)\end{aligned}$$

To complete the expression for $\Delta_{a,n}^2\delta$, the nutation increment has to be applied to δ' in (46). But this increment does not increase toward the pole, and may therefore be dropped.

164. It is important to know how near the pole these terms become important. For this purpose we remark that the values of An, B, C, and D are commonly less than $20''$, or $\cdot 0001$. The coefficients of the terms of the second order are therefore nearly always less than $0''\cdot 002$.

Near the pole, and at a distance of $p°$ from it, the values of $\tan\delta$ and $\sec\delta$ do not differ greatly from $\frac{57°}{p°}$. We therefore have the following approximations to values which the terms of the second order will never greatly exceed:

$$\left.\begin{array}{ll} p° = 12°; & \Delta^2 = 0''\cdot 01 \\ \text{,,} \quad 6 & \text{,,} \quad 0\cdot 02 \\ \text{,,} \quad 4 & \text{,,} \quad 0\cdot 03 \\ \text{,,} \quad 3 & \text{,,} \quad 0\cdot 04 \\ \text{,,} \quad 2 & \text{,,} \quad 0\cdot 06 \\ \text{,,} \quad 1 & \text{,,} \quad 0\cdot 12 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (49)$$

We conclude that when only the ordinary limit of precision, $\pm 0''\cdot 1$, is aimed at, the trigonometric reduction need be used only within 2° of the pole, but that, if the limit is to be $\pm 0''\cdot01$, either it or the development to terms of the second order should be used within 12° of the pole.

Section. IV. Construction of Tables of the Apparent Places of Stars.

165. The term *fundamental* is applied to a limited number of the best determined stars, the known positions of which are used as auxiliaries to determine the positions of all other heavenly bodies. There is no definable limit to the number of stars that may be used for this purpose. About the beginning of the 19th century Maskeleyne chose thirty-six of the brightest stars, nearly all of the first or second magnitude, scattered over that portion of the sky which could be seen at Greenwich, made frequent observations upon them, and thus determined their positions with all the accuracy of which his instruments permitted. These were used to determine the positions of other stars by methods the principles of which will be shown in the next chapter. Since his time the increasing requirements of astronomy have led to a continual increase in the number of stars regarded as fundamental.

In the preparation of each of the national astronomical ephemerides, a selection of stars to be regarded as fundamental has been made, and their apparent places given on the plan set forth in the preceding section. Quite independently of these lists, other lists, generally more numerous, have been prepared from time to time by astronomers for special purposes. The most complete list of this kind is found in the *Astronomical Papers of the American Ephemeris*, vol. viii., pp. 91-122. It comprises all the stars whose places are given in any of the astronomical ephemerides, with the addition of such other stars that the whole shall form a list scattered over the entire sky with as near an approach to uniformity as possible. The number of stars in this list is 1597. This number is more than double that thus far given in any one ephemeris. But it is not unlikely that at no distant date arrangements may be made between these publications which shall admit of the apparent places of the entire list of selected stars being regularly given.

When apparent places of such stars as these are required, not merely for a single year, but for a number of consecutive years, their computation can be facilitated by the use of suitable tables, by a method devised by Bessel and found in his *Tabulae Regiomontanae.* The method of constructing such tables will here be set forth in such a way that its application need not involve any difficulty to one acquainted with the subject.

166. The fundamental idea of the method is that the day numbers, omitting the minute terms of short period, are functions of the longitude of the moon's node and of the sun's longitude. They may, therefore, be divided into two parts depending on these two arguments.

The star numbers varying slowly from year to year, it follows that their products into the day numbers, the sum of which products is the reduction to apparent place, may be arranged, in the case of each star, into two tables, one depending on the node and the other on the sun's longitude, or the time of year. To show the process more in detail, we put

T_1, the time corresponding to the beginning of any solar year;

α_1, δ_1, the mean coordinates of the star for the beginning of that year;

τ, the fraction of the year after the beginning at which the apparent position is required.

Let us now see how the reduction from the mean place for T_1 to the apparent place for $T_1+\tau$ may be tabulated. By the developments in Chapter X., §§ 145, 146, the change due to precession and proper motion from the beginning of any solar year until the date τ may be expressed in the form

$$\left.\begin{aligned}\Delta_1\alpha &= \tau D_t\alpha + \tfrac{1}{2}\tau^2 D_t^2\alpha \\ \Delta_1\delta &= \tau D_t\delta + \tfrac{1}{2}\tau^2 D_t^2\delta\end{aligned}\right\} \qquad (50)$$

The speed of variation $D_t\alpha$ is computed numerically by the formulae of § 145. The term $D_t^2\alpha$ will never be required except when the star is near the pole. Whether it is used or not, the values of the two factors change from year to year only with great slowness. We may, therefore, compute their values numerically for some fundamental epoch, say 1900, and also their secular variations. The latter, at least in the case of $D_t^2\alpha$ and $D_t^2\delta$, may most easily be found by repeating the computation for a second epoch, say 1925 or 1950. In this way we shall derive, for each star, a general expression of the form

$$D_t\alpha = \alpha' + \alpha'' T$$

with similar expressions for $D_t^2\alpha$, $D_t\delta$, and $D_t^2\delta$. We then make a table of $\Delta_1\alpha$ and $\Delta_1\delta$ from (50) for intervals of ten sidereal days, or, if the star is near the pole, for every such day. If a ten-day interval is used, the increment of τ from one date to the next will be

$$\Delta\tau = \frac{10}{366{\cdot}24} = {\cdot}0273.$$

Starting the table from the beginning of the year, the successive values of τ and $\Delta_1\alpha$ will be

$$\left.\begin{array}{rccc} \tau = 0\,; & \Delta\tau\,; & 2\Delta\tau\ldots. \\ \Delta_1\alpha = 0\,; & \alpha'\Delta\tau\,; & 2\alpha'\Delta\tau\ldots. \\ \text{sec. var.} = 0\,; & \alpha''\Delta\tau\,; & 2\alpha''\Delta\tau\ldots. \end{array}\right\} \quad \ldots\ldots\ldots\ldots(51)$$

The products $\frac{1}{2}\tau^2 D_t^2\alpha$ may be included in $\Delta_1\alpha$ in the exceptional case when they are sensible. Thus, from a single table comprising an argument and two columns may be interpolated the value of $\Delta_1\alpha$ for 1900 and its secular variation for any date τ. By multiplying the secular variation by the fraction of a century elapsed since the fundamental epoch, and adding the product to $\Delta_1\alpha$ for 1900, the complete value of $\Delta_1\alpha$ for the year required will be obtained.

Of course the tabular value of $\Delta_1\delta$ may be formed in the same way.

Nutation. Passing to the nutation, we use the equations (3) of § 151. The four coefficients

$$\left.\begin{array}{l} \cos\epsilon + \sin\epsilon\sin\alpha\tan\delta \\ \cos\alpha\tan\delta, \\ \cos\alpha\sin\epsilon, \\ \sin\alpha, \end{array}\right\}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots(52)$$

are nearly constant for any one star, being, in fact, in the case of the second and fourth of these coefficients, identical with b and $-b'$, while the first and third are nearly equal to a and a', multiplied by constant factors.

The values of $\Delta\psi$ and $\Delta\epsilon$ by which these four coefficients are to be multiplied are found in § 134, where their terms are divided into three classes, which are to be tabulated separately as follows:

Terms depending on the node. Calling T_1 the time of beginning of the year, we compute the value of Ω, the longitude of the moon's node, for the five dates:

$$T_1; \quad T_1+100 \text{ sid. days}; \quad T_1+200 \text{ sid. days, etc.,}$$

or $$T_1; \quad T_1+0{\cdot}273; \quad T_1+0{\cdot}546; \quad T_1+0{\cdot}819; \quad T_1+1{\cdot}092,$$

which carries the computation past the end of the year. With these values of Ω the terms of $\Delta\psi$ and $\Delta\epsilon$ depending on Ω are computed and multiplied by the four corresponding numbers (52). This computation may readily be made for every year for which tables are to be used. We thus obtain so much of the reduction to apparent place as depends on the longitude of the node, which we may designate by the symbols $\Delta\Omega_\alpha$ and $\Delta\Omega_\delta$.

On Bessel's plan, the values of α_0 and δ_0 are added to these terms in the printed table, for which we then have five values for each year. From these the values for any intermediate date may be interpolated.

Annual terms. In these the nutation terms, which belong to the second class, are combined with the aberration, both being functions of the sun's true longitude. Since $\tau=0$ when the sun's mean longitude is 280°, it is easy to tabulate the values of $\odot$ for

$$\tau=0, \quad \tau=\Delta\tau, \quad \tau=2\Delta\tau, \text{ etc.,}$$

through the year. These values of $\odot$ are then used in computing the corresponding values of $\Delta\psi$ and $\Delta\epsilon$, and also of C and D from (15). By multiplying each by the proper coefficients, which are functions of the position of the star, so much of the reduction to apparent place as depends on the sun's longitude is thus tabulated as a function of τ. To the quantities thus tabulated are then added the values of $\Delta_1\alpha$ and $\Delta_1\delta$ for the corresponding values of τ from (51).

The secular variation of this part of the reduction may be computed and applied on the same principle with that of the precession.

From the four tables thus formed, two for each coordinate, we may form the apparent R.A. and Dec. of the star for any time τ in any year. We now have to show how the dates of the year are related to the meridian of the place for which an ephemeris of the star may be required.

167. Since the apparent places of fundamental stars are required almost entirely in connection with observations across the meridian of some observatory, the ephemerides give these places, not for mean noon, but for the moment of passage over the meridian of the principal observatory for which the ephemerides are constructed. Since the interval between two transits is a sidereal day, the tables are constructed for units of an integral number of sidereal days, and not for mean time. It follows that, for any one observatory and any one year, the factor of interpolation, omitting the entire days, will be the same for any star through the whole course of any one year. This factor depends on the relation of the beginning of the year to the meridian of the observatory in question.

The moment T_1 at which the solar year begins, being that at which the sun's mean R.A. is 18 h. 40 m., is a certain moment of absolute time, of which the expression in the local time of any place will depend on the longitude of that place. There will be one meridian, and no more, at which the sidereal time of T_1 will be 18 h. 40 m. This meridian is that which the mean sun crosses at the moment. It follows that this moment is mean noon on this meridian. Let us call the latter the *standard meridian* for the year. Let us put

k, the east longitude of the standard meridian from Greenwich.

It follows that the Greenwich mean time of the beginning of the solar year is $-k$ or 24 h. $-k$. This time may be computed from year to year by tables of the sun's mean longitude. In Appendix III. of the present work will be found a table of the times for the twentieth century. Changing the signs of these times, or subtracting them from one day, and converting the result into hours and minutes if required, we shall have the values of k. Next, let us put

λ, the west longitude from Greenwich of the meridian $\equiv M$ for which the ephemeris is required.

$\lambda + k$ will then be the distance west from the standard meridian to the meridian M in question. It follows that on this meridian we shall have at the moment of beginning of the solar year

$$\text{Local mean time} = 24\text{ h.} - (\lambda + k).$$
$$\text{Sidereal time} \quad = 18\text{ h. } 40\text{ m.} - (\lambda + k).$$

If we express λ and k in fractions of a day, the first transit of the vernal equinox over the meridian M in the course of any one year will occur at a sidereal interval 5 h. 20 m. $+ \lambda + k$ after the beginning of the year.

Also, the first transit of a star of R.A. $= \alpha$ over this meridian after the beginning of the year will follow the beginning by the sidereal interval

$$5\text{ h. } 20\text{ m.} + \alpha + \lambda + k - (24\text{ h. when necessary}).$$

This then will be the factor of interpolation for the first transit. The factor for any subsequent transit will be that corresponding to an integral number of days after this moment.

NOTES AND REFERENCES.

The large scale on which reductions from mean to apparent positions of stars, or the reverse, have to be made, has led to a number of methods and tables auxiliary to the regular ones of the ephemeris which alone have been treated in the present chapter.

Stone, E. J., *Tables for facilitating the computations of star constants* (Appendix to Cape Observations 1874), gives extended tables for the easy and rapid computation of the star constants, a, b, c, and d.

The Struve-Peters values of the astronomical constants were not introduced into the ephemerides until about 1850. In order to facilitate their use before that time the Poulkova Observatory computed and published tables of the day-numbers under the title *Tabulae Quantitatum Besselianarum* for the period 1750-1894. The later publications of the series differ from those given in the ephemerides by including the smaller terms of the nutation. The numbers are therefore given for every day. Before 1840 they are given only for every tenth day.

Auwers, *Tafeln zur Reduction von Fixstern-Beobachtungen für* 1726-1750 (Zweites Supplementheft zur *Vierteljahrsschrift der astronomischen Gesellschaft*, Jahrgang IV.), Engelmann, Leipzig, 1869, gives the day numbers for ten-day intervals with modern values of the constants, thus forming, with the Poulkova series, a complete series from 1726.

A modification of the independent day numbers, and of the method of using them, has been devised by Mr. W. H. Finlay, assistant at the Cape Observatory, which is believed to offer marked advantages over the usual ones, still tabulated in the ephemeris. The system is explained in *Monthly Notices of the Royal Astronomical Society*, vol. l., p. 497, (June 1890). Tables of the star constants used in this method have been published by the Cape Observatory, and the modified day numbers for use in connection with them have been published by the same institution since 1897.

Although the logarithms of the day numbers are given to four decimals in the ephemeris, three are practically sufficient in the reduction of meridian observed positions to mean place, unless near the pole. Tables of 3-place logarithms are found at the end of the present volume.

Several graphical systems for the reduction have been devised. One of these is by Mr. Finlay and is found, with the chart for its application, in *Monthly Notices, R.A.S.*, vol. lv., p. 15, (November 1894). Another by Erasmus D. Preston is found in *Bulletin of the Philosophical Society of Washington*, vol. iii., p. 182. This was independently distributed by the Society. It contains a diagram for finding the reduction graphically without the labour of computation.

The fact that a ten day ephemeris of a star for an entire year can be computed from tables of the form described in the present chapter in about an hour renders the use of such tables desirable in the computations of annual ephemerides extending through a number of years. Besides those of the *Tabulae Regiomontanae* which are based on the older values of the constants, tables for the fundamental stars by Leverrier are found in *Annals de l'Observatoire de Paris; Mémoires*, vol. ii. Tables for a larger number of stars, slightly different in some details, are found in *Star Tables of the American Ephemeris*, Nautical Almanac Office, Washington.

In the office of the British *Nautical Almanac* the reductions are understood to be computed for each separate day by the use of a "Star correction Facilitator," an ingenious instrument devised by Mr. T. C. Hudson, and described in the *Monthly Notices, R.A.S.*, liv., p. 90, (December, 1893).

CHAPTER XII.

METHOD OF DETERMINING THE POSITIONS OF STARS BY MERIDIAN OBSERVATIONS.

168. In this branch of practical astronomy everything relating to the management of the instrument, and the investigation of its performance, belongs to the subject of practical and instrumental astronomy, to be treated in another work. In the present work we shall develop the general principles which enter into the determination of positions of the fixed stars from observations.

Such determinations are divided into two classes, *fundamental* and *differential.*

Fundamental work consists in the determination of positions of fixed stars, the results of which are independent of any previous determinations.

Differential work is that in which positions previously determined are assumed as known, and new positions are fixed relatively to these assumed ones.

Even in fundamental work it is nearly always advisable, at least in the case of right ascensions, to assume certain positions of the stars in advance, with the view of subsequently correcting them from observations in such a way as to obtain results independent of any preceding work. Results of this kind are to be regarded as fundamental, leaving as differential only results in which preceding determinations are regarded as not subject to correction.

169. The ideal transit instrument.

The development of the subject requires a conception of the instruments used in determinations of position in the heavens, in a state of ideal perfection. The actual instruments used by the astronomer require a great number of small corrections for their errors and deviations. The object and result of these corrections is to reduce the results derived by the use of the instrument to what they would have been were the latter ideally perfect.

The instruments necessary for the determination of R.A.'s are the transit instrument and the sidereal clock, with their subsidiary appliances. That required for the declinations is a vertical circle. The transit instrument and the circle are commonly combined into a single instrument known as the *transit circle* or *meridian circle*; but, owing to their separate functions, they can be considered separately.

The ideal transit instrument is a telescope moving only on a fixed horizontal east and west axis at right angles to its line of sight, so that the latter describes the plane of the meridian. In the focus of its object glass a spider line is set at right angles both to the axis and the line of sight. The latter passes through the centre of the object glass and of the spider line. In the ideal instrument as described, the line of sight being always in the plane of the meridian, marks out the meridian on the celestial sphere. Observations with the actual instrument require a number of corrections for deviation from the ideal form. Everything relating to these corrections belongs to the subject of practical and instrumental astronomy, which is not treated in the present volume. What is essential for our present purpose is only the conception of the ideal instrument.

The ideal clock runs with perfect uniformity, so that the correction necessary to reduce the indication of its face to sidereal time is a quantity which varies uniformly with the time. This uniform variation, in the course of one day, is called the *rate* of the clock. The ideal rate can be determined by the difference between the clock times of transit of a star over the meridian on two consecutive days.

We have now to show how, with the ideal instrument as described, the right ascensions and declinations of stars are ideally determined.

Section I. Method of Determining Right Ascensions.

170. It will be most instructive to begin with the ideal case in which the right ascensions of all the stars are supposed to be unknown quantities, whose values are to be found by observation. Since the right ascensions are measured from the equinoxes, and since the latter is an imaginary point defined as that at which the sun apparently crosses the celestial equator, right ascensions must be determined by comparing the stars with the sun. We therefore observe the clock times of transit of the sun over the meridian with our ideal instrument day after day, through an entire year. We also observe the sun's declination at the same transits on a system which will be described in the next section.

We also observe the clock times of transit of a selected list of stars, preferably near the equator, through the seasons during which such observations can be made upon the star.

Assuming the obliquity of the ecliptic to be known, the longitude and R.A. of the sun on every day of observation can be computed from its observed declination by trigonometric formulae not necessary to be given here. It may be remarked that near the solstices, where the longitude and R.A. are near 90° or 270°, these coordinates cannot be determined accurately from the declination. But this is a practical detail which need not interfere with our ideal proceeding. Our results will depend upon observations of the sun's declination, made not too near the solstices. A small error in the adopted value of the obliquity will also be nearly or entirely eliminated, because it will have opposed effects at different seasons. Moreover, the obliquity itself may be determined from the observations of declination.

Let us put for the observations of any one day:

$T_{\odot}$, the clock time of transit of the sun;

$T_1, T_2, T_3, \ldots$, the clock times of transits of any number of stars on the same day;

$\alpha_\odot$, the right ascension of the sun, computed from the observed declination.

Since the sidereal time of transit of the sun is identical with its right ascension at the moment of transit, it follows that, if the sidereal clock is set exactly right, we should have

$$T_\odot = \alpha_\odot.$$

If, as practically is always the case, this equation is not exactly satisfied, the difference is the correction of the clock, which we call ΔC:

$$\Delta C = \alpha_\odot - T_\odot.$$

From repeated observations of the sun day after day, we have a series of values of ΔC. If the clock were correct and the observations without error, ΔC would vary by the same uniform quantity every day, and its general expression would be of the form,

$$\Delta C = \Delta C_0 + rt,$$

r being a constant expressing the daily rate of the clock. Practically, we have to suppose r a constant so long as no serious error will thus arise.

With the value of r and ΔC_0 the value of ΔC can be determined at the moment of transit of every star. The apparent R.A. of the stars observed will then be given by the equations

$$\alpha_1 = T_1 + \Delta C_1,$$
$$\alpha_2 = T_2 + \Delta C_2,$$
$$\cdots\cdots\cdots\cdots\cdots$$

On this system we may determine the R.A. of any number of stars as often as we please. Although, in the present state of astronomy, it is never necessary to adopt this ideal system, it is still true that the latter embodies the fundamental principles on which alone the absolute R.A.'s from the equinox can be determined. However complex the process may be, the R.A.'s of each star must ultimately depend upon a comparison with the sun, direct or indirect; and the R.A. of the sun must be regarded as a function of its observed declination. But, practically, this dependence of the stars upon the sun is brought about, not directly, but only indirectly, through correcting long series of observations so as to bring the results of observation into accordance.

171. Practical method of determining right ascensions.

It was pointed out in the preceding chapter that there is no immediate necessity for referring the R.A.'s of the stars to the actual equinox, and that, except for the necessity of comparing the R.A.'s and longitude, any other origin would serve the purpose as well as the equinox. But there is no visible point or system of points in the sky which can be used to define such an origin. The equinox, or as near an approximation to it as can practically be made, is therefore in permanent use.

A concession from rigour is, however, made by regarding as known quantities the R.A.'s of a system of fundamental stars extending round the circle of R.A., at not too great a distance from the equator, and using them to define the equinox.

The system now universally adopted is as follows:

Let us put:

$\alpha_1, \alpha_2, \ldots \alpha_n$, the adopted R.A.'s of the fundamental stars observed in the course of any one day or evening.

These R.A.'s may, in all ordinary cases, be taken from one of the ephemerides. The positions of the stars so used are practically more exact than any single observation that can be made upon them.

$T_1, T_2, T_3, \ldots T_n$, the clock times of transit of these stars.

Then, as in the case of the sun, a correction of the clock will be derived for each observation by subtracting T from α. The several values of the corresponding sidereal times, commonly taken to the tenth of an hour only, may be arranged in a table in the form shown in the following example, for which the numbers are derived from the Greenwich observations for 1901, January 4.

Royal Observatory, Greenwich, 1901, *Jan.* 4.

Star Observed.	Clock Time.	T. Seconds of Transit.	α. Seconds of R.A.	$\Delta C_{obs.}$	$\Delta C_{con.}$	$\alpha_{obs.}$	$\Delta\alpha$.
	h.	s.	s.	s.	s.	s.	s.
α Aquila	19·8	59·80	56·77	−3·03	−3·03	56·77	0·00
α Aquarii	22·0	44·95	41·98	2·97	3·05	41·90	−0·08
ι Pegasi	22·0	27·03	24·09	2·94	3·05	23·98	−0·11
ζ Pegasi	22·6	34·75	31·66	3·09	3·05	31·70	+0·04
μ Pegasi	22·8	16·76	13·69	3·07	3·05	13·71	+0·02
κ Piscium	23·4	54·86	51·87	2·99	3·06	51·80	−0·07
ι Piscium	23·6	55·02	51·97	3·05	3·06	51·96	−0·01
γ Pegasi	0·1	12·05	8·99	3·06	3·06	8·99	0·00
β Arietis	1·8	14·72	11·64	3·08	3·07	11·65	+0·01
α Arietis	2·0	40·12	37·04	3·08	3·08	37·04	0·00
σ Arietis	2·8	6·42	3·25	3·17	3·08	3·34	+0·09
α Ceti	3·0	10·96	7·88	3·08	3·08	7·88	0·00
δ Arietis	3·1	3·09	59·89	3·20	3·08	0·01	+0·12
ϵ Eridani	3·5	20·71	17·63	−3·08	−3·09	17·62	−0·01
Mean	0·3			−3·06			0·000

The column following the name of the star gives the clock time of transit over the meridian to the nearest tenth of an hour. This is practically the same as the hour and tenth of the star's R.A.

Column T gives the seconds and fractions of a second of clock time of transit as derived from the observation.

Column α gives the seconds of computed R.A. of the star as determined from the ephemeris, applying such small corrections as were deemed necessary.

The differences of these two numbers is the clock correction as derived from each separate star.

The mean of all the times in the second column is then taken and the means of all the observed clock corrections. It is assumed that the mean correction, −3·06 s., is the true correction of the clock at the mean of the clock times, or 0·3 h. sid. time.

By a comparison of the observations on the preceding and following days it was found that the daily rate of the clock was $-0{\cdot}18$ s.; with this correction and rate the concluded clock correction is found for each observation of the series by the equation

$$\Delta C = \Delta C_m - 0{\cdot}18t \text{ s.}$$

The values of the clock correction thus computed are found in the column following those observed. By applying them to the several clock times of transits, the R.A. of each star as derived from the observation is found, and its seconds given in the column $\alpha_{\text{obs.}}$.

The column $\Delta\alpha$ gives the correction to the adopted R.A. of each star as inferred from these observations. The concluded clock corrections are also applied to the times of transit of all the other stars, planets, and other bodies which may have been observed, and thus their apparent R.A.'s are derived.

At present, however, we are concerned only with the observations of the fundamental stars, and especially with the nature of the corrections derived in the preceding way. The main point is that the R.A.'s derived from the observations are not completely independent determinations, because that of each star is derived from the assumed R.A.'s of all the stars observed, itself included. It is evident that, on this system, the mean value of all the corrections $\Delta\alpha$ will vanish, or the mean of all the R.A.'s of a group of stars observed on any one day will come out the same as the mean of the adopted R.A.'s, some being increased and others diminished, so as to bring the whole in agreement among themselves. Hence, if the entire group is affected by any common error $\Delta\alpha$, the results of the observations will all be affected by this same error. In summing up the results for a year or a series of years, all the R.A.'s derived from observation, whether on the fundamental or other stars, will therefore be affected by a series of small errors

$$\Delta\alpha_1, \Delta\alpha_2, \Delta\alpha_3, \ldots,$$

which will be the mean errors of all the adopted R.A.'s of the individual groups of stars used in forming the clock correction during the entire period.

Entrance of systematic errors. In the case of any individual star the final error will be the mean of the errors of all the stars with which it was compared, these errors being weighted on a system to be explained presently. If all the errors in the adopted R.A.'s of the individual stars could be regarded as independent and accidental ones, each as likely to be positive as negative, the final results would be free from systematic error. But, as a matter of fact, the adopted R.A.'s may be affected by systematic errors of two kinds: one constant, the other varying with the R.A. It is evident that any systematic error in the observations of the sun may result in the adopted R.A.'s of the stars being measured from a point slightly different from the actual equinox. Such a displacement of the equinox will result in all the R.A.'s being in error by a quantity equal to that displacement. As already pointed out, this constant error is not of serious import unless in exceptional cases where ecliptic longitudes have to be used.

172. Elimination of systematic errors.

It is, however, of the first importance to eliminate systematic errors varying with the R.A. A study of the conditions of observation show how errors of this class may be perpetuated. Let S be any individual star and S_γ any fundamental star which has been used in deriving the clock error from which S is determined. Let N be the number of stars used on any one evening in determining these clock corrections, and let Δ_γ be the error in the adopted R.A. of S_γ. The R.A. of S resulting from the observation will in consequence of this error Δ_γ be affected by the error

$$\Delta\alpha = \Delta_\gamma \div N.$$

Taking for S_γ all the fundamental stars which have been used in determining S, we see that the latter will be affected by a certain mean error of all the fundamental R.A.'s used in determining the clock correction for S, these means being weighted in proportion to the number of times that each star S_γ was used for the clock correction. Now if, in the case of the star S, the stars S_γ were equally scattered all around the circle of R.A., the result for S would be affected only by the constant error common to all. But, as a matter of fact, observations are not ordinarily extended through more than a few hours of any one night, and, occasionally, a longer period during the day. The result will be that the error of S will not be the mean error of all the fundamental stars, but mainly of those which culminated within a short interval, say 2, 3, or 4 hours, of S. It follows that, if the values of $\Delta\alpha$ are systematically different in different hours of the circle of R.A., this error will be perpetuated with only a greater or less diminution. As the adopted R.A.'s are corrected from observations from time to time, the tendency will be to smooth off the systematic errors in question so that they shall approximate to the form

$$\Delta\alpha = a \cos \text{R.A.} + b \sin \text{R.A.}$$

That is to say, there will be a periodic error in the R.A.'s of all the stars which can be eliminated only by comparing the clock errors derived from stars as far apart as possible in R.A. This periodic error will be considered in a subsequent section. At present we pass to the constant error of the equinox, which we call E, the equinoxial error.

173. Reference to the sun—the equinoxial error.

We recall that the determination of this error must rest fundamentally upon observations of the sun. Ideally we have considered the R.A. of the sun as determined for each day. Practically, however, such a determination need not be made. The practical method consists in determining the error of the sun's tabular R.A. as found for every day of the year in the ephemeris, by systematic observations of the transit of the sun over the meridian, through considerable periods of time. What we then have to deal with is not the R.A.'s and Decs. of the sun as derived directly from observations, but small corrections to the values of these quantities as tabulated in the Ephemeris.

The steps of the process are as follows:

1. The R.A. and Dec. of the sun are observed on as many days as possible through the whole of one or more years, and the R.A.'s are reduced as if the sun were a star; that is, the clock correction used is that derived from the adopted R.A.'s of fundamental stars. All the observed R.A.'s of the sun will, therefore, be affected by the same equinoxial error as those of the stars.

2. Each R.A. and Dec. of the sun derived from the observations is compared with the positions of the sun given in the Ephemeris, and the difference taken. Leaving out accidental errors of observation, the residual differences between the observed and tabular positions are conceived to be due to three causes:

1. The equinoxial error.

2. A constant error in measuring all the declinations of the sun, which may arise from various causes.

3. The error of the obliquity of the ecliptic adopted in the ephemerides.

To show how these errors are determined, let us put:

λ, α, δ, the longitude, R.A., and Dec. of the sun at any moment;

ϵ, the obliquity of the ecliptic;

E, the equinoxial correction.

The declinations of the sun as given in the ephemeris are derived from the values of its longitude computed from tables of the sun's motion. From these longitudes the declinations are computed by formulae equivalent to

$$\sin \delta = \sin \epsilon \sin \lambda.$$

We have also the relations

$$\cos \lambda = \cos \alpha \cos \delta,$$
$$\cos \epsilon \sin \lambda = \sin \alpha \cos \delta.$$

From these equations we form equations of condition by the method set forth in the chapter on Least Squares. By differentiating the first equation and substituting the others we find

$$d\delta = \cos \alpha \sin \epsilon \, d\lambda + \sin \alpha \, d\epsilon.$$

If we put

$\Delta\lambda$, the correction to the longitude on any one day;

$\Delta\epsilon$, the correction to the obliquity of the ecliptic, which may be regarded as constant through the entire period;

Δ_0, a possible constant error in measuring all the declinations with the instrument;

$\Delta\delta$, the excess of the observed over the tabular declination: then each observation of declination will give the equation of condition:

$$\cos \alpha \sin \epsilon \, \Delta\lambda + \sin \alpha \, \Delta\epsilon + \Delta_0 = \Delta\delta.$$

If the tabular elements of the earth's orbit around the sun are correct, $\Delta\lambda$ is constant throughout the entire period of observation. In all probability the errors of the elements are so small that we may regard their possible effect upon the result as quite insignificant. Assuming $\Delta\lambda$ as a constant, we have to solve the above equations of condition by the method of least squares.

It is not, however, necessary to treat the corrections individually. The values of the coefficients $\sin\alpha$ and $\cos\alpha$ vary so slowly and regularly that we may use their mean values for each month as constant throughout the month. We then have twelve equations, one derived from the observations of each month. We may assign to these equations weights proportional to the number of observations. But, unless the latter are very unequally divided through the year (a circumstance which will greatly impair their value, and perhaps render them scarcely worth using), we shall get as good a result by assigning equal weights to the observations of each month as if we assigned weights dependent upon the number of observations. In fact, the errors which we have to fear are not the purely accidental errors, but possible constant errors continuing through a month, but varying from one month to another. Their possibility will lead us to diminish the weight assigned to a large number of observations in a single month, so as to make it approximate to the weights assigned to other months. As a general rule, the mean of the dates of observations in each month will not, in the general average, be greatly different from the middle of the month. We may, therefore, conceive the twelve monthly values of α to form a series scattered at equal intervals of 30° each around the circle. Thus we shall have twelve conditional equations:

$$\begin{aligned}
\cos\alpha_1 \sin\epsilon\,\Delta\lambda + \sin\alpha_1\Delta\epsilon + \Delta_0 &= \Delta\delta_1,\\
\cos\alpha_2 \sin\epsilon\,\Delta\lambda + \sin\alpha_2\Delta\epsilon + \Delta_0 &= \Delta\delta_2,\\
\cos\alpha_3 \sin\epsilon\,\Delta\lambda + \sin\alpha_3\Delta\epsilon + \Delta_0 &= \Delta\delta_3,\\
\vdots \qquad\qquad & \quad \vdots\\
\cos\alpha_{12} \sin\epsilon\,\Delta\lambda + \sin\alpha_{12}\Delta\epsilon + \Delta_0 &= \Delta\delta_{12}.
\end{aligned}$$

Proceeding now to the method of solution by least squares, the normal equation in $\Delta\lambda$ may be found by multiplying each equation by the coefficient $\cos\alpha$. It is true that the actual value of the coefficient of $\Delta\lambda$ is $\cos\alpha \sin\epsilon$, and if we multiply by this coefficient all our products will contain the common factor $\sin^2\epsilon$. But it will be more convenient to regard $\sin\epsilon\,\Delta\lambda$ as the unknown quantity, which we may call x. The general form of the equations will then be

$$\cos\alpha_i x + \sin\alpha_i \Delta\epsilon + \Delta_0 = \Delta\delta_i,$$

where $i = 1, 2, 3, \ldots 12$.

The normal equation in x derived by the method of § 36 will now be

$$\Sigma\cos^2\alpha \,.\, x + \Sigma\sin\alpha\cos\alpha \,.\, \Delta\epsilon + \Sigma\cos\alpha \,.\, \Delta_0 = \Sigma\cos\alpha\,\Delta\delta.$$

When the values of α are scattered equally around the circle, we have by known trigonometric theorems,

$$\Sigma\cos^2\alpha = 6,$$
$$\Sigma\sin\alpha\cos\alpha = 0,$$
$$\Sigma\cos\alpha = 0.$$

Thus our normal equation in x reduces to the very simple form

$$6x = \Sigma\cos\alpha\,\Delta\delta,$$

and the correction $\Delta\lambda$ is given in the form

$$\Delta\lambda = \frac{x}{\sin\epsilon} = \frac{\Sigma\cos\alpha\,\Delta\delta}{2{\cdot}40}.$$

Having thus found $\Delta\lambda$, we have next to determine its effect upon the R.A.'s. In the case of the sun, this coordinate, as tabulated in the ephemeris, is derived from the equation

$$\tan\alpha = \cos\epsilon\tan\lambda.$$

We have, by differentiation,

$$\sec^2\alpha\, d\alpha = \cos\epsilon\sec^2\lambda\, d\lambda - \sin\epsilon\tan\lambda\, d\epsilon.$$

We have also $\qquad \cos\alpha\cos\delta = \cos\lambda,$

whence $\qquad d\alpha = \cos\epsilon\sec^2\delta\, d\lambda - \cos\alpha\tan\delta\, d\epsilon.$

The mean value of $\cos\epsilon\sec^2\delta$ in the course of the year may be regarded as 1, and that of $\cos\alpha\tan\delta$ as 0. We may therefore put, as the mean result of the entire series of observations of the sun,

$$\Delta\alpha_\odot = \Delta\lambda,$$

which we regard as the definitive correction to the tabular R.A. of the sun.

We have also found a series of apparent corrections to the sun's tabular R.A. by the reduction of the observations in R.A. Let us put, for any day,

α comp., the tabular R.A. of the sun;

α obs., the R.A. derived from the observed times of transit, by applying the method of § 171, using the adopted R.A. of the fundamental stars.

Then, we put $$\Delta'\alpha_{\odot} = \alpha \text{ obs.} - \alpha \text{ comp.}$$

All these observed R.A.'s require the common correction E. Hence the actual correction to the tabular R.A. of the sun is

$$\Delta\alpha_{\odot} = E + \Delta'\alpha_{\odot}.$$

Equating this to the actual value of $\Delta\alpha_{\odot}$ found from the observed declinations, we have

$$E + \Delta'\alpha_{\odot} = \Delta\alpha_{\odot} = \Delta\lambda;$$
$$\therefore\ E = \Delta\lambda - \Delta'\alpha_{\odot},$$

in which we may use for $\Delta'\alpha_{\odot}$ its mean value for the entire series of observations.

It may seem that the number of quantities which we have had to change or drop in order to reduce our result to this simple form is so great that the errors thus arising may be important. But this will be the case only when the observations are very unequally scattered throughout the year. If there are an equal number of observations in every month, the normal equation will be found to reduce to this form; that is to say, the corrections $\Delta\epsilon$ and $\Delta\delta$ will be eliminated of themselves. If the observations are scattered very unevenly through the year, especially if comparatively few are made in some months, the equations of condition must be solved by assigning weights to the mean result for each month. The normal equation will then no longer reduce to the above simple form, and must be solved in the regular way.

The determination of the correction as here set forth comprises two steps, one the determination of the correction to the sun's absolute R.A. from the observations of its declination, the other that of the differences between the R.A.'s of the stars and of the sun. These two determinations may be considered as quite independent of each other. The equinox can be determined from observations of the declination alone made at some observatories and of R.A.'s alone at other observatories. Now that the change in the error of the sun's longitude in the course of a year is so small as to be completely masked in the errors of the observations, this method of independent determination of the two quantities is the better one to adopt.

Another consideration bearing on the case is that the personal equation of the observers in observations of the sun's limb is probably different from that for observations of the stars, and that this difference is far from being the same with different observers.

174. The general policy in the construction of the national Ephemerides has recently been, and still is, not to change the adopted equinox until a long series of observations at different observatories shall show a well-marked and undoubted correction to be necessary. The equinoxes now in use, that is to say, the mean R.A.'s of the fundamental stars, were determined in 1876 from all the best observations then available. The general mean result of recent observations seems to indicate a positive correction to the R.A. of all the stars; but, from the very nature of the case the results are somewhat discordant, and the amount of the correction is still doubtful. Its reality is yet more questionable in the light of the recently recognized "magnitude-equation" now known to affect the R.A.'s of all the stars observed up to the present time. The existence of this equation renders it probable that the general R.A.'s of the stars determined in past times may have been too small to an extent to more than neutralize the possible positive correction to the R.A.'s of all the stars, which we have mentioned as indicated by recent observations.

175. The Greenwich method.

The only observatory which, at the present time, makes it a point to determine the equinoxes every year from its own observations is that of Greenwich. Here a method devised by Airy is used, which, though involving the general principles just set forth, deviates in detail.

The division of the equinoxial correction into two parts, the one applicable to the sun's tabular R.A., the other to the differences between the R.A.'s of the sun and stars, is not recognized. The method consists in taking the apparent corrections to the sun's R.A. and Dec. obtained in the usual way, and converting them into errors of the tabular longitude and latitude. The combined effect of the two errors $\Delta\alpha_{\odot}$, and $\Delta'\alpha_{\odot}$, is to produce in the errors of latitude an annual period of the form

$$\Delta\beta = E \operatorname{cosec} \epsilon \cos \odot.$$

The error $\Delta\epsilon$ of the obliquity produces in $\Delta\beta$ a term of the form

$$\Delta\beta = \Delta\epsilon \sin \odot,$$

while there may be a constant error in all the measures of declination made with the instrument in the course of the year.

The mean of all the observed errors of the latitude during each month, gives an equation of the form

$$a + b \cos \odot + c \sin \odot = \Delta\beta,$$

and the solution of all the equations thus formed gives a, b, and c. Then, $E = b \operatorname{cosec} \epsilon$, $\Delta\epsilon = c$. The result thus reached is doubtless the same as if the method of the present chapter were applied. But the method does not separate the two parts of which the correction E is composed.

Section II. The Determination of Declinations.

176. The ideal meridian circle.

The development of the principles on which declinations of stars are determined requires a statement of the fundamental idea of a meridian circle. The essential parts of this instrument are a finely graduated circle rigidly attached to a telescope (see Fig. 36). The latter is, in principle, the transit instrument already described, and therefore has but one motion, that in the plane of the meridian on a fixed horizontal east and west axis. The plane of the circle is parallel to the tube of the telescope, and in the plane of the meridian. At the eye end, in the focus, is a horizontal spider line, at right angles to the vertical line over which transits are observed. The result of these rigid connections is, that when the instrument is turned on its axis the angular motion of the line of sight is equal to the angle through which the circle has turned. We have to show how this angle is measured.

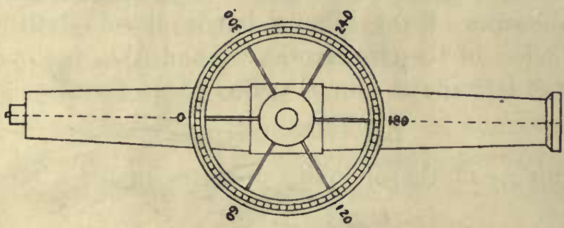

FIG. 36.

The circumference of the circle is divided into 360 parts of one degree each by fine lines or graduations, and each of these is subdivided into equal parts, generally 2′ or 5′ each. The graduations are numbered from 0° to 360°, and are visible through at least one pair of microscopes at opposite ends of a diameter. These microscopes are firmly fixed to the supporting pier, and therefore do not revolve with the instrument. For our present purpose, we need consider only a single microscope. This is supplied with a micrometer, by means of which the position of any graduation in the field of view of the microscope may be accurately measured.

The result of this combination is that the varying position of the circle becomes a continuous quantity, that is, motions ever so small may be measured. Suppose, to fix the ideas, that the graduation 28° 16′ is exactly in a certain part of the field of the microscope, which we take as the zero point. Then we say that the circle-reading is 28° 16′ 0″·0. Then, give the circle a small motion forward. If we find that the micrometer, when set on the graduation 28° 16′, now reads 7″·5, we say that the circle-reading is 28° 16′ 7″·5, and we know that the circle has been moved through an angle of 7″·5.

It follows that, corresponding to each position of the circle, and therefore to each direction of the line of sight of the telescope, there is a certain circle-reading. A fundamental principle of the method is, that a single reading tells us nothing of the absolute direction of the line of sight; but that the difference between two readings is equal to the arc through which the circle and telescope have turned between the readings.

In an instrument of perfect stability the circle reading should always remain the same for the same direction of the telescope. The direction might be the zenith, the nadir, the equator, or the pole. But, as a matter of fact, the reading for any fixed point changes by minute amounts not only from day to day, but even through different hours of the day. The determination of these changes forms one of the most troublesome problems with which the observer has to deal. We shall begin by ignoring them, and showing how positions of the stars are determined, supposing the instrument stable.

177. Principles of measurement.

Supposing our instrument ideally perfect, which, in practice, it never is, we have to show how fundamental declinations are measured with it. In doing this it will be convenient to replace the declination by the polar distance, from which we can, at any time, pass to declination by the simple process of subtraction from 90°. The polar distance of a star being defined as its angular distance from the North Pole, its determination would be extremely simple if only the pole were a visible point in the heavens. We should set the instrument on the pole and determine the circle reading $\equiv C_p$. We then should point the instrument at a star as it passes the meridian, and call the circle reading C_s. The difference, $C_s - C_p$, corrected for refraction, is the polar distance of the star as given by the instrument, and $90° - (C_s - C_p)$ is its declination.

The pole, not being a visible point in the heavens, has to be otherwise defined. Its true position, being the line of the instantaneous axis of rotation of the earth, is midway between the points at which a star near the pole crosses the meridian at an upper and lower culmination. It follows that if we put C_u for the circle reading when a circumpolar star crosses the meridian above the pole, and C_l for the reading when it passes below the pole twelve hours later, we shall have

$$C_p = \tfrac{1}{2}(C_u + C_l),$$

always supposing the zero-point constant during the twelve hours.

With this value of C_p the polar distance and declination of any star will be given by the preceding equations. No fundamental determinations of declinations can be made except in this way.

If the instrument were perfectly stable, that is to say, if the circle reading, when the telescope is set on the pole, were the same for a whole year, the essential principles of the method as thus set forth would be complete. But, as a matter of fact, the reading C_p may change from day to day, or even from hour to hour. This renders it necessary to have some fixed point of reference, the absolute position of which is arbitrary, but which can be determined at any time. Let N be the circle reading for such a point, which we call the zero-point, and which we suppose to be determined as often as necessary. Then making the observations above described, we have only to substitute $C-N$ for C; in other words, we may subtract the reading N for the moment of observation from all the observed circle readings on the stars, and use the difference instead of C.

More specifically, let us suppose that, at three different times, we observe

A circumpolar star at upper transit;

The same star at lower transit;

Any other star, S.

As before, let the three circle readings for these stars be

$$C_u,\ C_l, \text{ and } C_s.$$

Also let the circle readings for the zero-point at the three times be

$$N_1,\ N_2, \text{ and } N_3.$$

We then put
$$C'_u = C_u - N_1,$$
$$C'_l = C_l - N_2,$$
$$C'_s = C_s - N_3,$$
and we shall have
$$\text{Polar Distance of Star} = C'_s - \tfrac{1}{2}(C'_u + C'_l).$$

Through this process results will be the same as if the stability of the instrument were perfect.

For the zero-point in question the nadir is now almost universally used. It is determined by pointing the telescope vertically downward at the surface of a basin of quicksilver, and applying certain devices by which the verticality of the line of sight may be ascertained. The details of the method belong to the subject of instrumental astronomy, and cannot be entered upon here. For our present purpose, the nadir is simply a fixed direction for which the circle reading may be determined at any time.

In the preceding outline we have left out of consideration the various corrections due to precession, nutation, refraction, instrumental errors, and other causes, in order to facilitate the reader's grasp of the essential process. The latter results in independent determinations of the absolute declinations of the stars, substantially the same as if the pole were a visible point at which the instrument could be pointed at any time. This is the ideal result at which work with an instrument should aim.

178. Differential determinations of declination.

We now pass to differential determinations. In these the polar point and the fixed N-point are determined from the declinations of fundamental stars, assumed to be known in advance. Let us put

δ, the known or assumed declination of such a star;

C_s, the circle reading when the instrument is set on this star;

C_{eq}, the circle reading when the instrument is pointed at the equator;

C_p, the reading for the pole.

Since the arc from the star to the equator is equal to the declination of the star, it follows that, having made the observation of C_s, we infer that, if the instrument were pointed at the celestial equator, the circle reading would be

$$C_{eq} = C_s + \delta,$$

and for the pole, $$C_p = C_s + \delta - 90°.$$

If C'_s is the reading for any other star, and we put δ' for its declination, we shall have

$$\delta' = C_{eq} - C'_s.$$

In practice a number of standard stars are observed in the course of an evening, from each of which a value of C_{eq} is derived. The mean of these values is the value of C_{eq}, with which the declinations of all the other stars are determined by the above formula.

179. Systematic errors of the method.

It will be seen that this method is analogous to that applied in the case of R.A.'s. There is, however, an important difference in the nature of the systematic errors to which the method is liable in the two cases. The stars in any one region of the heavens, or in any hour of R.A., culminate in the course of a year at every hour of the day in succession. Consequently, systematic errors arising from diurnal changes of temperature and all other causes which go through their period in the course of a day are in great part eliminated from observations extending through an entire year. In other words, so far as unavoidable systematic errors of observations are concerned, all the stars are, so to speak, on the same footing.

But this is not the case with the declinations. Since any star always culminates at nearly the same altitude, any systematic error depending upon the zenith distance will repeat itself indefinitely. It is found from experience that the declinations of stars given by different instruments show very appreciable systematic differences. In good instruments the difference rarely, if ever, amounts to a second of arc, but may be an important fraction of a second. In the imperfect instrument of former times it may have been greater than a second.

The result is that, if we compare the equatorial or polar point of the instrument derived from a group of stars in one declination, it may be systematically different from that derived from a group in a very different declination. If both groups are combined, the result will be a heterogeneity in the declinations finally derived, which may seriously detract from their usefulness. Although the practical methods of dealing with this case are not strictly germane to the present work, it is necessary to show their general character, in order to be able to deal in the most intelligent way with the results actually found in published catalogues of stars prepared from the work of different observatories. The principal reason for using the differential instead of the absolute method in declinations is the avoidance of the labour of repeated determinations of the nadir point with an instrument which is probably not stable through any one day. By determining the equatorial or polar point from fundamental stars, this labour is avoided and more observations can be made. Such systematic errors as may affect the result need be no greater than those of the fundamental stars themselves. With a fairly stable instrument it is possible to arrange observations so that the determination of the nadir point will not be really necessary even if fundamental results are aimed at. There are two methods of doing this.

The first method is to adopt as fundamental stars only stars quite near the pole, say within 10° or 12° of that point. The systematic errors in the declinations of stars are smaller the nearer they are to the pole; and within the distance above mentioned they may be regarded as unimportant for ordinary purposes. Moreover, their effect will be almost entirely eliminated if the stars are observed both above and below the pole, as they pass the meridian at the upper and lower culmination. In this way the polar reading of the instrument may be determined from each night's work, and determinations of declination, practically absolute, may be made at all declinations.

The same result will be reached if the adopted declinations are made to agree with those determined by the instrument itself either by the above method or by the absolute method.

It is, therefore, not necessary to follow the method rigorously year after year. If, by adopting it, corrections are found for a limited number of fundamental stars, and the corrected values of the latter are then used for the equatorial or polar point, it will not be necessary to continue the observations of the polar stars.

The second method consists in using as fundamental stars only those included in some one zone of declination, 5° or 10° in breadth. The systematic differences within such a zone may probably be regarded as evanescent. If the stars to be determined also lie within the same zone, we shall have a set of declinations affected only by the common error of the fundamental declinations in the zone, which error we conceive to be capable of ulterior determination and call h.

If, following this method, stars scattered at widely different declinations in the sky are observed, all the resulting declinations observed above the pole will be in error by h, and those below the pole by $-h$. There will therefore be a systematic difference of $2h$ between the declinations of the same star when observed above and below the pole, which will show the value of h. This being determined, all the declinations can be corrected by this amount, and thus reduced to the values which the instrument itself would have given had it been used as a fundamental one.

When neither of these methods is applied, and when stars of different declinations have been used without any discussion of the systematic discordances between the results derived from them, the results cannot be regarded as fundamental. Nor can their accuracy be estimated except by a comparison with other authorities.

CHAPTER XIII.

METHODS OF DERIVING THE POSITIONS AND PROPER MOTIONS OF THE STARS FROM PUBLISHED RESULTS OF OBSERVATION.

Section I. Historical Review.

180. The Greenwich Observations.

The material at the command of the astronomer for the determination of positions of the stars consists mainly in catalogues of such positions at various epochs, as derived from observations of right ascension and declination made at observatories during the past two centuries. Owing to the diversity in the construction of the instruments of observation, in the method of using them, and in the adopted systems of deriving and publishing their results, the material in question is so heterogeneous that few features are common to all the catalogues. The method of utilizing it therefore requires a special study of each individual catalogue, as well as a comprehensive idea of the nature of the observations on which all the results depend. The mastery of the subject will be facilitated by beginning with a brief outline of the labours which modern astronomers have undertaken for the purpose in question.

Notwithstanding the imperfections of his instruments, the catalogue of stars constructed by Flamsteed, first Astronomer Royal, during the few years preceding his death, which occurred in 1719, was a great improvement on any preceding work of the kind, forming in a certain sense the basis of Bradley's work half a century later, as well as determining its direction. The most familiar remnant of Flamsteed's work is the system of numbers attached to his catalogue of stars, which are still used to designate such of the stars catalogued by him as are not designated by a letter on Bayer's system. But observations of the accuracy necessary to fixing the proper motions of the stars began with Bradley, Astronomer Royal of England, in the middle of the eighteenth century. Previous to his time, 1750 to 1756, the instruments and methods of determination were so imperfect that it is now possible, from the data which have since accumulated, to compute positions of the stars at any previous epoch with a higher degree of accuracy than the astronomers of the time were able to observe them. Notwithstanding the excellence of Bradley's observations, his instrument for measuring declinations was of the older kind.

It was not till half a century after his time that the advantages of using a complete circle, graduated through its entire circumference, for the purpose of measuring declinations was fully understood by astronomers. Down to nearly the beginning of the nineteenth century the mural quadrant was the principal instrument for this purpose. As implied by its name, this instrument consisted of a quadrant, the actual arc of which was, however, somewhat more than 90°, attached to the face of a wall in the plane of the meridian. The telescope moved on a centre coincident with that of the graduated arc. It is readily seen that, how great soever might be the care and skill employed in the construction, the readings of the arc were subject to errors arising from the unavoidable non-coincidence of the centre of motion of the telescope with the geometric centre of the quadrant —to errors of graduation—and to changes in the position of the instrument from time to time, due to slow motions of the supporting wall. Moreover, observations during any one period could be made only on one side of the zenith. It was therefore necessary, when observations were to be made on the other side, to take the quadrant down and remount it.

The telescope of the quadrant being supposed to move in the plane of the meridian, sometimes served the purpose of a transit instrument for the observation of right ascensions. The quadrant with its telescope thus served in a rude way the purpose of the modern meridian circle. The use of a separate transit instrument made its way very slowly, not being introduced at the Paris observatory until the beginning of the nineteenth century. Bradley's observations in right ascensions were made with this instrument, a circumstance to which their superiority is largely due.

Bradley's instruments, improved though they were, continued to be used at Greenwich until 1812-16, when the mural quadrant was replaced by a mural circle. The work with this instrument by Pond, Astronomer Royal 1812-1835, was far superior to any that had preceded it. He added a second mural circle, and made observations with the two conjointly, so as to obtain the supposed benefits of a combination. His observations with these circles have in recent times been partly reduced by S. C. Chandler, who found them to be of a degree of excellence, especially as regards their freedom from systematic errors, that has rarely been exceeded since his time. It is therefore to be regretted that, with the exception of a few fundamental stars discussed by Chandler, no results of Pond's work on the stars are yet available except those computed by himself, and therefore derived by the imperfect methods then in use.

Pond's successor in the office of Astronomer Royal was George Biddle Airy, who held that position from 1836 to 1881, a period of forty-five years. Airy's abilities as a planner and administrator of work were of the highest order. His system was based on the idea that one directing head could work out all the formulae and prepare all the instructions required to keep a large body of observers and computers employed in making and reducing astronomical observations. A few able lieutenants, who would see that all the details were properly carried out, were an adjunct of his system. Acting on these ideas, he reduced the work of the observatory to a system more comprehensive in its details than anyone had ever before attempted in the conduct of astronomical operations. The main object he had in view was the determining of positions of the heavenly bodies. He adopted the system of collecting the results of the observations of the stars from time to time in catalogues, each embracing several years' work, generally between six and ten. The transit instrument and mural circle, both excellent of their kind, were continued in use until 1850, when the great transit circle, devised in all its details by Airy, was installed, and still remains the principal meridian instrument of the observatory.

This instrument has proved one of the most useful in all star determinations except those special ones demanding the highest degree of delicacy. Its construction is an interesting reflection of Airy's methods. He thought out what he might well suppose to be all possible sources of systematic and accidental error, devised means of avoiding or eliminating them, established a complete system of supervision, and then assumed that the results of his system could be regarded as absolute determinations, which needed little further investigation so far as possible corrections to their results were concerned. Acting on this system, his transit circle was not reversible, a quality necessary only when it is assumed that an instrument may give different results if the pivots are turned end for end in their bearings. The circle is not adjustable on the axis, because the errors of graduation were determined once for all, and were not supposed subject to any further correction. No allowance was made for possible systematic errors in determining the line of collimation of the telescope. In a word, the idea that the instrument might well be subject to errors arising from obscure or unknown causes, and that it was desirable to vary in every possible way the methods of using it in order to test the existence of such errors, was not so prominent in his mind as it was in that of the school of Bessel.

One result of this is that the results with this instrument require corrections for systematic errors of various kinds. But another fortunate result is that the determination of these corrections is always possible, and when observations are made on a system so nearly uniform through long periods of time, the task of determining and applying such corrections is much easier than when the plan of work is frequently changed. With all its shortcomings, the Airy transit circle has proved to be the most serviceable meridian instrument ever constructed. The result is that the Greenwich observations during the past half century afford the broadest basis we now possess for the determination of those stars of which accurate positions are most required.

181. The German school.

Contemporaneous with the accession of Pond to the Directorship of the Greenwich Observatory was the foundation by Friedrich Wilhelm Bessel of the German school of practical astronomy. The fundamental idea of this school in the trial of its instrument reverses the maxim of English criminal law. The instrument is indicted as it were for every possible fault, and is not exonerated till it has proved itself correct in every point. The methods of determining the possible errors of an instrument were developed by Bessel with an ingenuity and precision of geometric method never before applied to such problems. Not only this, but even when every source of error admitting of determination and correction has been allowed for, the instrumental arrangement must admit of being varied from time to time in order that, if any undiscovered errors still exist, they may be detected by the discrepancies between different methods of observation.

Bessel's fundamental instrument, after a few years' work with an old circle and transit, was a meridian circle constructed by Reichenbach. Although this instrument approached the modern form, its construction was far from perfect, and, so far as precision of individual results is concerned, it is not likely that the work of Bessel with it was really superior to that of Pond. The far greater place which it has filled in the progress of practical astronomy is due more to the excellence of Bessel's system of reduction and discussion than to the precision of the instruments and observations themselves. He also, as compared with the Astronomers Royal at Greenwich, laboured under the disadvantage of commanding only the slender means of an ill-endowed observatory, itself only an adjunct to a university, as compared with the resources of an institution conducted under the auspices of a great government.

182. The Poulkova Observatory.

Bessel's ablest contemporary, imbued with a like spirit, and working much on the same system, was Friedrich Wilhelm Struve, then Director of the Observatory of Dorpat. It became his good fortune to secure the support of a powerful government in putting into practice the methods of the German school. About 1835 he secured from the Emperor Nicolas of Russia full authority to erect and equip an astronomical observatory of the first class, which should be a credit to the empire to which it belonged. The new establishment was erected on a slight eminence near the village of Poulkova, 18 kilometres south of the gate of St. Petersburg. Struve's special purpose was to introduce a new era into astronomical determinations by combining, on a large scale, the qualities of the most refined instruments that art could make with the skill of the most capable observers. It was found that when the latter intelligently devoted the most painstaking attention to the avoidance of every source of error, a degree of excellence in their work was reached which was not possible with mere routine observers.

In one important point he replaced the system of Bessel by an older one. For the determination of the positions of the fundamental stars, which was one of his first objects, he did not depend upon the meridian circle, by which observations in both coordinates are made simultaneously, but constructed the transit instrument and vertical circle as two separate instruments. The vertical circle was especially designed to measure zenith distances, not only at the moment of passing the meridian, but within a short distance of it on each side. In the hands of C. A. F. Peters one observation with this instrument was worth as much as twenty, thirty, or even forty made by routine observers with the meridian circle.

The results of the fundamental determinations of star places at Poulkova have been published at intervals of twenty years, the epochs of the catalogues having been 1845, 1865, and 1885.

Latterly the number of stars to which attention is devoted at Poulkova has been largely increased, one of the works being a redetermination of the positions of the stars observed by Bradley, more than three thousand in number.

The two great observatories of Greenwich and Poulkova, through their rich resources, the excellence of their instruments, and the permanence of their policy, have taken the leading place in supplying material for the fundamental data of astronomy. The other national observatories, as well as several university observatories, have however made, from time to time, valuable contributions toward the same end. Among these will be found the national observatories at Berlin, Paris, and Washington, and the observatories of the universities of Strassburg, Abo, Dorpat, Cambridge, Edinburgh, Glasgow, and Oxford, as well as those of the leading American Universities.

There are other observatories whose energies have been directed rather to the numerous telescopic stars than to the brighter fundamental stars. Here the Paris Observatory is worthy of especial mention, as well as several university establishments, some of whose publications are cited in the notes at the end of the present chapter.

183. Observatories of the southern hemisphere.

We have thus far glanced only at the observatories of the northern hemisphere. One result of this hemisphere having been the first seat of civilization is that the material available for star determinations in the southern celestial hemisphere is much more scanty than for the northern. The observatories of Europe, being generally between 45° and 60° of latitude, have not been able to advantageously extend their observations to more than 30° of south declination. Poulkova, in latitude 60°, is most unfavourably situated in this respect. The parallel of 30° south is on its horizon, so that the sun at the winter solstice culminates at an altitude of less than 7°. The general rule is that, near the horizon, vapours in the atmosphere detract from the accuracy of observations. The probable error of any astronomical observation continually increases from the region around the zenith toward the horizon,—the increase being very slow down to an altitude of, say, 30°,—but being more and more rapid from that point to the horizon itself. The general rule is that very little weight can be assigned to observations of position at altitudes less than 10°. The Poulkova observations may be regarded as somewhat exceptional in this respect, since the unusual clearness of its atmosphere permits observations to be advantageously extended nearer to the horizon than is possible in less favoured regions.*

Before the foundation of the Cape Observatory the observations on the southern stars were made almost entirely by individuals at establishments more or less temporary. The first enterprise of this kind was that of Lacaille, who was sent by the French authorities to the Cape of Good Hope on an expedition for astronomical purposes in 1750. During the two following years he made observations on the positions of nearly ten thousand stars with a very primitive telescope, used to make observations on zones of stars with the instrument in a fixed position for each zone. As Lacaille had no declination micrometer, it was not possible to determine zenith distances in the usual way by a horizontal thread, and the ingenious device of a rhomboid was adopted. This consisted of four strips of metal placed diagonally in the field, so that the times of entrance of the stars into the rhomboid and of their exit from it could both be observed and recorded. The mean of the two times was that of the transit across, the middle vertical line forming one diagonal of the rhomboid, while the difference of the times showed the distance of a star above or below the horizontal diagonal.

Naturally, the degree of precision reached by this method was quite low, but the work has served a very useful purpose during the century and a half which has elapsed since its completion. About the middle of the last century a reduction of Lacaille's observations was made by Baily under the auspices of the

* The author was once informed by Otto Struve, then Director at Poulkova, that during the Crimean War he could see through the great telescope the men on the decks of the British fleet, lying off Kronstadt, at a distance of some 25 miles.

British Association for the Advancement of Science, which published the results in a catalogue.

The next attack on the southern hemisphere in the same direction was made in 1822-26, under the auspices of General Brisbane, Governor of New South Wales, by Rumker. Some years later the observations were reduced and a catalogue of the results published by William Richardson, whose name the work commonly bears. This catalogue includes 7835 stars.

Shortly afterward Lieut. Johnson, at St. Helena, made a comparatively good series of determinations on 600 of the brighter stars.

The observatory of the Cape of Good Hope was established in 1830. Since that time it has been the chief centre of activity in the direction now under consideration. Its instrumental means and the methods of applying them have continually been improved, until, under the direction of Sir David Gill, who took charge of it in 1880, its work stands second in excellence to that of no observatory in the world.

184. Miscellaneous observations.

The work at Greenwich and Poulkova is, so far as positions of the fundamental stars are concerned, pre-eminent for its systematic character and for the long period through which it has extended. But many important though more temporary works have been carried out for the same purpose. First in the order of time must be mentioned that of Piazzi at Palermo, who, before and after the beginning of the nineteenth century, made a long series of observations on the fundamental stars, employing the transit instrument and the complete circle. His declinations of the fundamental stars were almost or quite the first made with the latter instrument. The observations were not, however, confined to the fundamental stars, but included nearly ten thousand stars of all the brighter magnitudes. Unfortunately only the imperfect reductions of Piazzi himself have as yet been available for the use of the astronomer; but a reduction with modern data and by modern methods is now being executed under the auspices of the Carnegie Institution by Dr. Herman S. Davis.

The works of the same class by Struve at Dorpat, before his removal to Poulkova, Argelander at Abo, Airy and his successors at Cambridge, various observers at Washington, as well as numerous lesser works of various observers will be cited hereafter.

185. Observations of miscellaneous stars.

The observations thus far reviewed are mainly those leading to determinations of the positions of fundamental stars. As already remarked, the distinction between stars which are fundamental and those which are not is somewhat vague. Yet, a fairly definite line may be drawn between observations leading to independent and accurate determinations of a limited number of stars, and those in which precision is sacrificed in order to extend the work to a larger number of minute stars. Observations of the former class have been mainly of the kind which in the preceding chapter were defined as fundamental, those of the latter class as differential. It may also be remarked that the determinations of the first class have been mostly confined to a few thousand of the brighter stars, including the most of those to the sixth magnitude, and occasionally a few of the seventh or eighth.

During the time of Bradley, and of his immediate successors, no attempt was made to determine the positions of the innumerable faint telescopic stars which stud the heavens, nor even to catalogue them. The first serious work in this direction was that of Lalande at Paris, who, near the close of the eighteenth century, and therefore contemporaneously with Piazzi, undertook the work of cataloguing all the stars visible in his instrument with a completeness which has scarcely been exceeded even up to the present day. His *Histoire Céleste*, which is in use even now in determining the proper motions of the fainter stars, is a long enduring monument to his industry. His energies were so entirely absorbed in the work of observation that he made no serious attempt at their complete systematic reduction. This much needed work was carried out by Francis Baily of England in 1840, and published in a thick volume containing the reduced positions of the stars observed by Lalande. But the data and methods of reduction then available were both imperfect, and the need of a complete rereduction has not yet been supplied. The best approach to it is found in a set of tables by von Asten, by which the astronomer may rapidly reduce any of Lalande's observations with data, which, although better than those used by Baily, do not fully satisfy modern requirements.

When the object is to determine the positions of the greatest possible number of stars, the observations have to be made by zones of declination, of greater or less breadth, according to the requirements of the case. The advantage of the zone system is that the observer does not have to move his telescope through wide arcs of declination between one observation and the next, but keeps it in nearly the same position during a period of several hours, perhaps through a whole night's work. The breadth of the zone depends upon the number of stars which it is desired to include. Sometimes the system has been to keep the telescope in a fixed position through the entire course of an evening, observing as many as possible of the stars which pass over the threads in the focus. As the field of view cannot advantageously take in more than a fraction of a degree, this system is a very slow one if it is intended to cover the entire sky. The more usual proceeding has, therefore, been to include a zone a few degrees wide, generally less than 5°, in each night's work.

Lalande's observations were made on this system. Twenty years later zone observations covering a considerable portion of the northern sky were commenced by Bessel at Königsberg and Argelander at Abo. The results of these two works have been collected by Weisse and Oeltzen in well-known catalogues.

In 1846 a series of zone observations, intended to ultimately cover as much as possible of the sky, was commenced at the Naval Observatory at Washington. But the work was left unfinished after two or three years. The observations have been published, but not in complete form. Some of them labour under the disadvantage of having been made by inexperienced observers who made many errors in writing down the numbers of their records, while others are of the best class.

About 1850 Bond, at the Harvard Observatory, observed a few zones near the equator, which are quite unique in astronomy. They were made with the great equatorial, which was clamped in a fixed position for each strip observed, and the R.A.'s and Decs. of the stars determined as they passed through the field. The results, which are found in the early volumes of the Harvard Annals, extend to fainter stars than have since been catalogued or even listed. They will, however, be included in the International Chart of the heavens now being made by photography at a large number of observatories in both hemispheres. Probably all those actually observed by Bond will be in the equatorial zone, which is being photographed at the Observatory of Algiers.

When Le Verrier took charge of the Paris Observatory in 1853, one of the projects which he instituted was that of the redetermination of all of Lalande's stars. This work was completed, and the results have been published during the past ten years in a catalogue filling four large quarto volumes. This immense work includes, not only the fainter stars observed by Lalande, but nearly all the stars of the brighter classes. But the method of reduction adopted by Le Verrier, and pursued since his time at Paris, is far from the best. Not only is all the work purely differential, both in R.A. and Dec., but sufficient attention has not been paid to the sources of systematic error to which such work is liable, especially in the adopted positions of the standard stars. The result is that, at the best, it is scarcely possible to apply any systematic corrections which will not leave accidental errors outstanding larger than should be found in results of the highest class. This does not greatly diminish the value of the work for the faint Lalande stars, but does for the brighter stars.

In 1865 the *Astronomische Gesellschaft*, an international association, having its headquarters in Germany, formed the design of a complete redetermination of the positions of all the stars in the northern celestial hemisphere, those within

10° of the pole excepted, down to the ninth magnitude, with as near an approach as possible to the modern standard of precision. With a single gap, the results of this work have all been published in the several catalogues issued by the society.

When the observations for this work were approaching completion, the project of extending it to 23° south declination was undertaken, and catalogues down to this point are now in process of preparation and publication.

Durchmusterungen. The star-lists, familiarly known as "Durchmusterungen," belong to a different class from the preceding, not being intended to record the accurate position of a star, but only its approximate position with sufficient precision to enable the star to be certainly identified. The first comprehensive work of this sort was carried on at Bonn by Argelander and Schönfeld, and extended from the North Pole to $-1°$ of south declination.

Schönfeld afterwards extended the work to $-23°$. From this parallel to the south pole a Durchmusterung is being carried on in the Argentine Republic, at the Cordova Observatory, by Thome, who has published three volumes of it, extending to $-53°$ of declination.

The *Cape Photographic Durchmusterung* by Gill and Kapteyn, is based on photographs of the sky taken at the observatory of the Cape of Good Hope, and covers the sky from $-18°$ to the South Pole. This is the best arranged and digested work of its class that has yet appeared. The number of stars included in it is not however so great as in the Cordova work, and the magnitudes assigned to the stars are subject to revision.

Section II. Reduction of Catalogue Positions of Stars to a Homogeneous System.

186. Systematic differences between catalogues.

The leading observatories at which meridian observations are made collect the results from time to time in the form of catalogues of the positions of stars. Generally such a catalogue is given in connection with the observations of each year. After a number of years the annual results thus derived may be combined in a single catalogue, in which the mean positions of the stars are reduced to some common equinox of reference. Positions of the brighter and more important stars are generally contained in quite a number of these catalogues, and when the definitive position of a star is to be worked out, the best result is reached by combining the data of all the catalogues in which it occurs.

When we compare the mean position of a star for any one epoch, as found in different catalogues, we are to expect differences arising from the accidental and unavoidable errors of observation, which it is desirable to eliminate by combining as many authorities as possible. If we represent by δ_1, δ_2, δ_3 ... the differences between the coordinates of the same stars in one catalogue A, and in another catalogue B, the mean of these differences should, if they were wholly in the nature of independent and fortuitous errors, converge toward zero as their number increases. But, as a matter of fact, this is seldom found to be the case. To fix the ideas, let us suppose a comparison to be made between all the declinations in a zone 5° wide. We shall nearly always find that the mean value of the differences, as the number of comparisons is increased, converges toward some well-marked positive or negative value, and not towards zero, as it should if the errors of each authority were purely accidental. in comparing even the best catalogues this value may be several tenths of a second. This shows that, in addition to the accidental errors necessarily affecting all astronomical determinations, there must be some source of systematic error affecting the positions in one catalogue differently from those in another. The possible sources of such errors are many. They may be classified as follows:

1. *Differences in the methods and data of reduction.* Examples of this class are differences in the adopted value of the constants of nutation and aberration; differences in the adopted positions of the fundamental stars used for clock errors; differences in the constant of refraction; the employment of an erroneous latitude

or non-correspondence of the adopted latitude with that given by the nadir point of the instrument. All differences of this class admit of being reconciled by applying to the positions given in the catalogues the corrections necessary to reduce the results to what they would have been had the same data and methods of reduction been used in each. The extent to which it is advisable to apply these corrections must depend on the labour involved and value of the results to be reached.

2. *Causes of systematic error which we know may have affected the observations, but of which we have no way of estimating the magnitude.* One example of this arises where the errors of graduation of a circle have not been investigated at all, or have not been well determined. In such cases all the declinations may be in error by amounts varying with the zenith distance; but it will be impossible to determine the amount except by comparison with other authorities. Another example is the possible difference between day and night observations, arising from the different conditions under which they are made. We may take it for granted that the diurnal changes of temperature of an instrument and its surroundings may vary the adjustments in a way not admitting of determination, yet following a general law which may be expressed in a formula of which the numerical elements can be inferred from a comparison of authorities. The personal equation of an observer may also be different for stars observed by day and by night. In this case it will be possible to formulate some sort of a law which the errors should follow; but the determination of the exact amount of the error will not be possible except by a comparison of results.

3. *Errors arising from unknown causes which elude both investigation and exact statement.* We find by comparing the independent results of the work of different observatories that differences show themselves which we may attribute to a number of causes, temporary or variable, the results of which do not admit of being reduced to any well-defined law. The causes of the errors thus indicated may be temporary,—varying from one night to another—or they may affect the work of an

observer or an instrument through months or years. They can be determined in each case only by an intercomparison of independent authorities.

Until recent times little or no account was taken of these systematic differences among catalogues of stars. When the position and proper motion of a star were to be determined from a combination of observations or of catalogue positions, the deviations of the several catalogues from the general mean were regarded as purely accidental errors. The practice of correcting catalogues for systematic differences among them is largely due to the example and researches of Auwers, which date from 1865. In explaining the system of correction we must premise that the processes are largely tentative, and that, from the very nature of the case, the exact values of the corrections must remain more or less in doubt.

187. Systematic corrections to catalogue positions.

The general idea on which systematic corrections of the kind in question are to be derived is that, in each individual catalogue, whole groups of stars may be affected by common sources of error peculiar to the group. For example, there are many causes which may affect all stars of the same declination with one and the same constant error. Other causes may affect all the stars of one Right Ascension in the same way, but stars of different Right Ascensions in different ways. Apart from errors in the elements of reduction, refraction, nutation, aberration, etc., these errors can be brought out only by a comparison of separate and independent authorities *inter se*. The idea is then to apply to positions given by each separate authority such corrections for the several groups of stars as shall bring the whole into general harmony. In doing this different weights will be assigned to different authorities according to the supposed freedom of the results from sources of systematic error. When the positions of each authority are thus corrected the discrepancies still outstanding should follow the law of purely fortuitous errors.

It should be remarked that the relative weights assigned to the different authorities for the purpose of this combination may be very different from those to which the positions of individual stars are entitled after the corrections have been made. For example, so far as freedom from systematic errors of Class *B* is concerned, the positions of stars obtained at Greenwich and at Poulkova may be entitled to equal weight. But the probable deviation of one observation of declination at Greenwich is more than double that of one observed at Poulkova. In consequence the weight to be assigned to a single observation at Poulkova is more than four times that assigned at Greenwich.

When a system of corrections is constructed for a number of standard catalogues, the harmonious set of positions to which all the catalogues are reduced is known as a *fundamental system*. Such a system may be considered as embodied in a more or less complete set of corrections of the kind described, or in a standard catalogue of stars constructed from all available observations after systematic corrections have been applied to the individual authorities.

The purpose being to find, not corrections to individual stars, but the mean corrections to whole groups of stars, two methods may be adopted. One method consists in comparing the positions of standard stars in each separate catalogue with some one standard catalogue. It is not necessary, in the first place, that this standard of comparison should be a definitive one, because the correction of the standard itself will always be in view. Let us suppose, to fix the ideas, that the adopted standard positions of the fundamental stars in a certain zone of declination, 5° or 10° in breadth, are compared with those found for the same epoch from all the other available catalogues. Let the mean deviations of the standard thus found be:

From the standard itself	zero
„ Catalogue *A*	δ_1
„ „ *B*	δ_2
....	

Then, by applying the several corrections δ_1, δ_2, etc., to the coordinates in the several catalogues, they will all be brought into harmony with the standard.

But the standard itself may need correction, because the proper standard is the general mean of all. To find this mean let the weights assigned to the several catalogues be

$$w_0, w_1, w_2, \ldots,$$

w_0 being the weight of the provisional standard itself. Then, the weighted mean

$$\Delta = \frac{w_0\delta + w_1\delta_1 + w_2\delta_2 + \ldots}{w_0 + w_1 + w_2 + \ldots},$$

will be the deviation of the standard from the mean of all the authorities, itself included. Since this last mean is the final standard, it follows that Δ will be the deviation of S from this final standard, and $-\Delta$ the correction to reduce the standard to the general mean. The corrections then necessary to reduce the several catalogues to the mean standard are:

Standard catalogue ;	corr.	$= -\Delta.$
A ;	„	$= \delta_1 - \Delta.$
B ;	„	$= \delta_2 - \Delta.$

....

the weighted mean of which is zero, as it should be.

The preceding method, if applied without modification, is subject to the drawback that it is not easy to find any one catalogue sufficiently complete and comprehensive to serve as the sole basis of comparison with all other catalogues at all epochs. The method may, therefore, be modified, as necessary, by comparing pairs of catalogues A and B for the same epoch. If δ_1 and δ_2 be the unknown corrections of the two catalogues for any zone or region, and $\Delta_{1,2}$ their mean difference within this zone or region we shall have

$$\delta_1 - \delta_2 = \Delta_{1,2}.$$

We may then reduce all the comparisons of either of the two catalogues with the standard $\equiv S$ to the mean of the other, and thus gain for each the benefit of the comparisons of S with the other. For example, assume the case that we have 6 stars common to A and S, and 4 common to B and S; but a much greater number common to A and B. From the comparison of A and B we find a value of $\Delta_{1,2}=A-B$. Let us also put

ϵ_1, the mean of the 6 differences $S-A$;

ϵ_2, the mean of the 4 differences $S-B$.

We shall then have, for $\delta_1=S-A$, two independent values, namely

1. That from direct comparison: $\delta_1=\epsilon_1$.
2. That through B: $\delta_1=\epsilon_2+\Delta_{1,2}$.

The concluded values of δ_1 and δ_2 are then the weighted mean of these two values, giving

$$\delta_1=\frac{6\epsilon_1+4(\epsilon_2+\Delta_{1,2})}{10}.$$

In a similar way we have for δ_2 the two values ϵ_2 and $\epsilon_1-\Delta_{1,2}$, giving

$$\delta_2=\frac{6(\epsilon_1-\Delta_{1,2})+4\epsilon_2}{10}.$$

188. Form of the systematic corrections.

We have now to consider the general form of the systematic corrections ordinarily applied. It is common to regard them as two in number for each coordinate, one being a function of the Right Ascension alone; the other of the Declination alone. The designations of the corrections are as follows:

$\Delta\alpha_\alpha$, correction to the R.A. depending on the R.A.;
$\Delta\alpha_\delta$, correction to the R.A. depending on the Declination;
$\Delta\delta_\alpha$, correction to the Declination depending on the R.A.;
$\Delta\delta_\delta$, correction to the Declination depending on the Dec.

The necessity for the correction $\Delta\alpha_\alpha$ has arisen through the wide adoption of an erroneous system of Right Ascensions dating from the time of Pond, which were transplanted by Le Verrier in his fundamental catalogue used in reducing the Paris observations, and used by Airy in the early Greenwich work. By successive revisions of the fundamental Right Ascensions this error may be gradually reduced, the rapidity of the reduction depending on the number of hours through which observations extend during each night. What remains of the error is in this way smoothed off to the general form

$$\Delta\alpha_a = a \cos \text{R.A.} + b \sin \text{R.A.}$$

In all good catalogues it should be assumed as of this form, if it exists at all. As a matter of fact, however, it has been nearly or quite smoothed out in the recent Greenwich catalogues, and in most other modern catalogues, except those based on the Paris Right Ascensions.

It may be remarked that Auwers does not assume the correction to be of this form, but determines it from hour to hour, smoothing off the results so that they shall be represented by a regular curve.

The error which $\Delta\alpha_\delta$ is intended to eliminate has arisen mainly from the pivots of the transit instrument not being perfect cylinders, whereby the line of collimation slightly deviates to one side or the other of the meridian of the instrument. An erroneous determination of the error of collimation will also produce an error of this sort. Near the pole other causes, due to personal equation, come into play. The personal equation of the observer is very likely to be different for a slow moving star than for the rapidly moving equatorial stars. Moreover, close circumpolar stars can be better observed by eye and ear than by the chronograph. But when the latter is used for the quick moving stars, while those near the pole are observed by eye and ear, there is likely to be a change in the personal equation from one class to the other.

The Right Ascensions of the catalogue may, as already shown, also require a constant equinoctial correction. This may be combined either with $\Delta\alpha_a$ or $\Delta\alpha_\delta$. The former is the more common method, since the correction is in this way very easy to determine. Auwers, however, adds it to the values of $\Delta\alpha_\delta$, which on the score of exactness is the preferable system.

189. Method of finding the corrections.

The practical method of forming the tables for $\Delta\alpha_\delta$ and $\Delta\delta_\delta$ is to begin the comparison by zones of declination. A zone of declination of suitable breadth, generally 5° or 10°, is taken, and for all the stars within it common to the two catalogues the differences between the fundamental catalogue and that to be corrected are formed for each coordinate, and the mean taken. This mean will be the preliminary value of $\Delta\alpha_\delta$ or $\Delta\delta_\delta$ for the middle declination of the zone. This process being extended to all the zones in the catalogue, we shall have values of the correction for each zone.

These corrections are then to be arranged in a column and smoothed off, so as to form the ordinates of a regularly varying curve, and interpolated to every round 5° or 10° of declination, as the case may be, with the declination as an argument.

To form the corrections depending on the R.A., the catalogue places should first be corrected for the terms depending on the Dec., and the corrected places compared with those of the standard catalogue. The mean of the residual differences is then taken for every 1^h, 2^h, or 3^h, of R.A., smoothed off, and arranged in a table with the R.A. as the argument.

In the formation of $\Delta\alpha_a$ it is advisable to give most weight to stars within 30° of the equator, and little or no weight to stars north of 50°, or perhaps 60°. The exact method of proceeding must depend on the peculiarities of the catalogue to be reduced, especially on the methods of observation and reduction.

190. Distinction of systematic from fortuitous differences.

An important question in preparing such tables is how far the differences which we find should be regarded as accidental rather than systematic and should, therefore, be ignored as arising from fortuitous errors. By the theory of these errors, when the mean of a number of them is taken, the probable value of this mean will diminish as the square root of the number whose mean is taken. We may therefore say, in a general way, that if the mean systematic correction thus derived does not exceed the probable mean of the fortuitous deviations, we should disregard it. The same may be true even should the mean value be greater than that set by the limit. That is to say, if we put

ϵ, the general mean value of the differences whose mean is taken;

N, the number of these differences;

Then, if the final mean is less than $\frac{\epsilon}{\sqrt{N}}$, there is no reason to regard the differences as systematic. And even should it exceed this limit, the reality of the difference may be in doubt. In any case we should see whether the differences remain of the same sign through several zones of declination. If they do, the reason for considering them real is strengthened, if not it is weakened. In view of the probable amount of the accidental deviations, the rule should be, when in doubt as to the amount of a correction, to assign it the smallest probable value.

From this point of view it seems quite likely that many of the systematic corrections found in existing tables should be regarded either as unreal or as being too large. It may also well be that they vary too rapidly from one zone of Declination, or hour of Right Ascension to another.

Practically, it will seldom be necessary to construct new tables or corrections for any of the fundamental catalogues, because such tables have already been constructed by Auwers, Boss and the present writer, and can be used to such an extent, or combined in such a way as the computer deems best. Every new catalogue that appears will, however, need examination with a view of constructing tables for the purpose in question, and it is in this case that the preceding methods and principles have to be applied.

191. Existing fundamental systems.

A "fundamental system" of star-positions may be defined in either of two ways. One is by a sufficiently extensive catalogue of fundamental stars, in which the position and proper motion of each individual star has been worked out with the greatest possible precision. It is then assumed that the errors of such a catalogue, both accidental and systematical, are as small as possible, and that the latter vary very slowly from one region of the sky to another.

A fundamental system may also be defined by tables of corrections of the form just explained for the best existing star catalogues. For by applying these corrections to catalogue positions of the stars, positions and proper motions of each individual star are obtained which will be in harmony with the fundamental system on which the corrections are based.

The following are the fundamental systems which have been most used.

1. The system constructed by Auwers for use in reducing the observations of zones of stars made under the auspices of the *Astronomische Gesellschaft.* It is, therefore, commonly known as the "A.G. System." The star positions which form it are found in No. xiv. of the *Publicationen der Astronomischen Gesellschaft,* and an extension into the southern hemisphere in Publication xvii. of the same series.

In this catalogue the modern positions, generally for a mean epoch about 1865, were derived from a careful discussion and combination of all the best modern determinations. The proper motions, which are an essential part of any fundamental system, were found by a comparison of the observations of Bradley (mean epoch about 1755) with a preliminary (not with the definitive) catalogue of modern positions, and were not further altered.

The result of this is that, if the fundamental places are reduced back to 1755, they will not rigorously agree with Auwers-Bradley, but will deviate by an amount equal to the correction applied to reduce the modern provisional place to the definitive place.

It is now well established that the Bradley positions are affected by considerable systematic errors. The consequence of this is that, through the proper motions, the A.G. system is, for our epoch, affected by systematic errors of the opposite algebraic sign, increasing uniformly with the time.

2. At nearly the same time as the A.G. system appeared the system of Professor Lewis Boss. This included only the declinations. The proper motions were derived by the rigorous process of a least square solution of all the results of observations.

This system was adopted in the American Ephemeris from 1883 to 1899, and was employed in the researches made in the office of that work up to 1897.

The declinations were so thoroughly worked up by Boss that the continued use of the A.G. system of declinations until 1900 is to be regarded as unfortunate.

3. The third system is that of Newcomb, found in the *Astronomical Papers of the American Ephemeris*, vol. viii. In forming the fundamental declinations the processes show the following main features.

(*a*) The systematic corrections to the Boss system were found in advance of determining places of the individual stars.

(*b*) In the case of each instrument used in forming these systematic corrections, the error of its declinations of stars near the equator was determined by the general principle that the planets move around the sun in great circles. In consequence, the declinations of the planets are, in the general mean, to be regarded as absolutely correct. Accordingly, if the declinations of any planet, through an entire revolution, are found, by instrumental measurement or by comparison with the declinations of a fundamental catalogue, to be in error by the constant quantity ϵ, we conclude that this error is not real, but is due to an error in the instrument or the catalogue and correct its results accordingly.

(*c*) When circumpolar stars are observed both above and below the pole, the systematic error must be zero at the pole.

(*d*) The polar correction being 0, and the equatorial correction ϵ, it is assumed, in the case of each good instrument or catalogue, that the error varies uniformly between these limits.

(*e*) South of the equator the error, in the absence of any means of determining it, was assumed to be nearly constant.

The positions of this system have been used in all the national Ephemerides except that of Germany from 1901 to the present time. They were introduced into the American Ephemeris from 1900.

4. *The new system of the Berliner Jahrbuch, by Auwers.* During the years 1895-1903 Auwers was engaged in a thorough reconstruction of the A.G. system, resulting in a new fundamental catalogue. The catalogue itself has not yet appeared,

but the corrections to the A.G. positions and proper motions of the stars, which will suffice for constructing it, are found in the *Astronomischen Nachrichten*, vol. 164. The work of Auwers not being published in all its details, a description of his methods cannot yet be given.

5. *The new Boss system.* At the same time that Auwers was carrying on his work, Boss was also constructing a new fundamental catalogue. The resulting places of the stars of the southern hemisphere appeared in 1898, and will be found in the *Astronomical Journal*, vol. xix. The positions of the northern stars were completely worked out in 1903, and the results will be found in vol. xxiii. of the same publication.

The question of the systematic differences between the last three-named systems is of interest. Practically, they may be regarded as identical so far as the Right Ascensions are concerned. This identity arises from the fact that all three catalogues were based on the same adopted position of the equinox, and that the systematic errors of observation in Right Ascension, which we may suppose to arise from the diurnal changes of temperature, are largely eliminated in the course of a year's work with a good instrument. There will naturally be small differences of the form $\Delta\alpha_\delta$, but these prove not to be great except near the pole, where determinations are necessarily a little indefinite, owing to uncertainty as to the personal equation of observers.

In the case of the declinations, the differences, though small, are well marked. Near the pole all three authorities agree, as they should, because all systematic errors of good determinations are small. But, from 20° polar distance to the equator, Auwers places the stars a little farther south than Newcomb, and Boss farther south than Auwers. Near the equator, where the difference is a maximum, the mean corrections to the declinations of Newcomb, as found by the two authorities, are:

$$\text{Auwers, Corr.} = -0''\cdot13.$$
$$\text{Boss, Corr.} = -0''\cdot28.$$
$$\text{Whence Boss} - \text{Auwers} = -0''\cdot15.$$

These differences will increase very slowly owing to corresponding differences in the proper motions. From such re-examination as the author has been able to give to the subject, the presumption seems to be that his system does really require a correction in the direction indicated by Boss and Auwers, and the probability is that the truth lies somewhere between these two authorities. The difference of 0″·15 between them is too small to be of serious import for the present.

In connection with all three catalogues are given tables of systematic corrections of the form already described for the positions of all the principal catalogues of stars.

Section. III. Methods of Combining Star Catalogues.

192. Use of star-catalogues.

We have shown in Chapters XV. and XVI. how, when the mean position of a star at some epoch is given, and its proper motion, its apparent position at any time may be found. We have now to show how these fundamental elements are derived by combining the data given in various star catalogues.

The term *catalogue of precision* is applied to those catalogues of which the purpose is to give precise positions of the stars. The designation is used in contradistinction to the lists which are intended only to enable the stars to be identified, or in which precision is sacrificed to number.

An independent catalogue is one in which the positions are derived solely from a limited number of observations made at some one observatory. There are also catalogues which give positions of stars based on a combination of the work of various observatories, with a view of deriving as accurate results as possible, but at present we are not concerned with these.

The results and data usually given in a catalogue are as follows:

1. The R.A. and Dec. of each star, as derived from all the observations, and referred to the mean equator and equinox of some convenient epoch.

2. The mean date of all the observations on the star.

3. The number of observations on which the place depends.

4. The precessions in R.A. and Dec. for the epoch of reference.

5. The secular variation of the precessions. In most independent catalogues, this refers to the precession alone, not to the annual variation.

6. The proper motion of each star, when obtainable.

It should be added that, in independent catalogues, the proper motions are added merely for the convenience of the astronomer using the work, and cannot in rigour be considered as belonging to it, because they cannot be based on the same observations as the positions of the catalogue itself.

In recent catalogues the beginning of the solar year is adopted as the epoch of reference. But, in former years, the distinction of the civil from the solar year was little attended to, and the equinox is frequently called that of January 1, although, quite likely, the beginning of the solar year was actually used.

The relation of the epoch of reference to the mean date of the observations must be understood. If, in constructing the catalogue, the observed positions were reduced to the epoch of reference by precession, aberration, and nutation alone, without applying a correction for proper motion, the given position will be that for the mean date, though the equator and equinox will be those of the common chosen epoch. This is theoretically the best course. In many catalogues, however, the proper motion for the interval between the date and the epoch, as well as the precession, is applied in the reduction, with a view of giving the actual position of the star at the same epoch as that of the equinox of reference. The user of the catalogue should always know which system is adopted, and use the results accordingly.

The problem of deriving the position of a star for any date from a modern catalogue of precision requires only the application of formulae and methods already developed in the chapters on the reduction of positions of the fixed stars. The principal question left open will be whether to reduce the mean place from the epoch of the catalogue to the required epoch by means of the precession, proper motion, and secular variation found in the catalogue, or by the trigonometric method. The choice will depend on the length of the interval and the declination of the star. As a rough and ready rule, it may be said that, if the product of the interval in years by the secant of the declination exceeds 40, the trigonometric method should be adopted; but, if less than 40, the development in powers of t may be used, if it is found more convenient. But this would prescribe the trigonometric method of reducing stars within 1° 30′ of the pole, even through a single year, where it would not be necessary. The limiting value of the product $(T-T')\sec\delta$ may, therefore, be carried up to 50, or even 100 or more, near the pole.

In ordinary astronomical practice, the position of a star found in this way from any standard catalogue, or in any of the modern independent catalogues, will be precise enough for general use. The problem we have to consider is that of deriving the position and proper motion of a star with the highest attainable precision from a combination of all the independent catalogues in which it is found.

193. Preliminary reductions.

Having found the star in any catalogue, certain preliminary steps will be required to reduce the data to the required form. These are:

1. *Possible reduction for proper motion.* As already mentioned, each catalogue has two epochs: one the mean epoch of all the observations, the other the epoch of the equinox of reference. It should be understood that the latter epoch really has nothing to do with time, because it merely defines the particular system of coordinates to which the position is referred. Time enters only as the simplest method of defining the direction of the fundamental axes of reference; the problems growing out of this direction are purely geometric.

When a uniformly varying quantity is observed at several dates, and the mean of all the results taken, this mean is the most probable value for the mean date of all the observations, irrespective of the rate of variation. It follows that the ideally proper position to give in a catalogue is the mean of the observed positions referred to such equinox of reference as may have been selected.

When the observed position is reduced from the mean date of the observing to the date of the equinox of reference, by applying the adopted motion during the interval, the position is no longer, rigorously speaking, an observed one, but one in which observation and the reduction for proper motion are combined. It follows that, if the computer desires to follow a rigorous system, he should, in all catalogues where the reduction in question is made, free the given place of the star from this correction. Although this modification is seldom of practical importance in the work, the habit of adopting rigorous methods in astronomy cannot be too highly recommended. If the reduction is not made, the given position will be regarded as if observed at the epoch of the equinox of reference.

2. *Systematic corrections.* The next step is to apply such systematic corrections to the coordinates of the star as may be required to reduce them to a homogeneous system. The method of deriving these corrections has already been set forth; but the computer will rarely have to do this, as the three sets of existing tables of corrections are sufficiently accurate for all practical purposes. In making a choice of or a combination among the authorities, it might be a good practical rule to prefer the correction of the smallest absolute amount, because, as already pointed out, the probability is that such a correction will be too large. When so large that it cannot be doubted, it indicates some source of systematic errors in the catalogue which should diminish the weight assigned, and therefore the effect of the correction upon the final result. If the catalogue is one for which no table of corrections is found, one may easily be constructed by the methods of the last chapter.

3. *Assignment of weights.* The next step will be to assign a weight to the catalogue position thus corrected. Were all observations completely independent determinations, and equally good, the weights in each case would be proportional to their

number. But each instrument has peculiarities of its own, in virtue of which the determinations of any one star with it may be affected by a constant error, which will be less the better the instrument. Although this constant error may, if considerable, be diminished by the systematic corrections, it will never be reduced to zero. We are, therefore, to consider that the probable error ϵ of a position taken from any catalogue is determined by the equation

$$\epsilon^2 = \epsilon_0^{\ 2} + \frac{\epsilon_1^{\ 2}}{n},$$

ϵ_0 being the probable amount of the constant error, and ϵ_1 that of the varying accidental error. The weights are then taken so as to be inversely proportional to ϵ^2, the general form being

$$w = \frac{ne^2}{n\epsilon_0^{\ 2} + \epsilon_1^{\ 2}},$$

where e is the probable error chosen to correspond to the unit of weight, and n the number of observations.

There are, of course, great diversities in the precision of the observations on which various catalogues depend. This must be taken account of in assigning the weights.

Careful investigations of the data on which the various catalogues have been constructed with a view of expressing the weight as a function of the number of the observations have been made in connection with the three fundamental catalogues already described. It will hardly be worth while in the case of any existing catalogue for a computer to reinvestigate this subject for himself. He can either adopt one of the latest tables, or combine any two or all three according to his judgment.

The result of the three processes will be, in the case of any one catalogue, that:

At a certain epoch t (that of the mean of all the observations), the right Ascension or Declination of the star, referred to the mean equator and equinox of an epoch T, had a certain value α or δ, and that this value is entitled to a certain weight w.

The star being found in as many good catalogues as it is thought worth while to use, the results for the different places of the stars will then be that:

$$\left.\begin{array}{ll}\text{At the epochs} & t_1,\ t_2,\ t_3, \dots\ t_n \\ \text{For the equinoxes of} & T_1,\ T_2,\ T_3, \dots\ T_n \\ \text{The R.A. or Dec. was} & \alpha_1,\ \alpha_2,\ \alpha_3, \dots\ \alpha_n \\ \text{With the respective weights} & w_1,\ w_2,\ w_3, \dots\ w_n \end{array}\right\} \dots\dots\dots\dots (a)$$

194. The two methods of combination.

The problem now is, from these data to derive the most likely mean position at some chosen epoch T_0, and the proper motion. There are two ways of doing this, of which the first is simplest in principle, but not always the easiest in practice:

1. Each of the positions $\alpha_1, \alpha_2, \dots$ may be separately reduced to the mean equinox of the chosen epoch T_0 by precession alone, using the trigonometric method when advisable. We shall then have a series of values of α which, were the position of the star on the sphere invariable, and the catalogue places perfect, should all be identical. Differences among these numbers arise from errors of the catalogue places, and from the proper motion of the star. The best position and proper motion can then be determined by the method of §§ 39-41.

2. With an approximate position α_0 for any date and an assumed proper motion μ_0 the position of the star may be computed for the several epochs T_1, T_2, etc., and compared with the positions given in or derived from the catalogue. The excess of each catalogue position over the computed position gives a correction to the latter for the mean date of the catalogue; and from the combination of all these corrections the most likely corrections to α_0 and μ_0 may be derived by a least-square solution.

It will be seen that the fundamental difference between the two methods is that, in using the first, we reduce each observed place to the initial epoch, while in using the second we reduce an assumed place for the initial or some other epoch to the date of each observed place.

This second method is preferable in the case of those fundamental and other stars for which positions for various dates derived from a single value each of α_0 and μ_0 are available. Otherwise method 1 is preferable.

195. Development of first method.

The epoch T_0 of the equator and equinox to which the positions are to be reduced must first be decided on. If a general catalogue for astronomical researches based on past as well as present observations is in view, the most convenient epoch will probably be 1875, as this was in extensive use during the last quarter of the nineteenth century, and is that to which the catalogues of the *Astronomische Gesellschaft* are reduced. But, if the positions are required only for current use, it will be better to choose 1900, or even some later epoch, according to the requirements.

All the positions are then to be reduced from their several equinoxes T_1, T_2, etc., to the selected equinox T_0, by precession alone. For all the remoter epochs this is to be done trigonometrically. But when the interval of reduction is short, and the star not near the pole, it may be found most convenient to use the annual precessions for the two epochs, or that for the middle epoch, or that for any date not too remote from the epoch, combined with the secular variation. When the latter is used the proper motion should, in rigour, be omitted in computing it; but commonly the effect of including it will be so slight that its retention or omission will be unimportant.

It can very seldom be worth while to compute the secular variations for this express purpose. With the aid of the tables given in Appendix IV. the trigonometric reduction is so easy that it may involve less labour to use it, even for an interval as short as ten years, than it will to compute and apply the annual precessions and secular variations.

When, as is always the case in modern catalogues, the precessions for the date of the catalogue are given, these may be used, care being taken to first reduce them to one and the same standard value of the precessional constant.* Using precessions, some one of the following formulae may be applied. Put

* Tables for reducing Struve's precession to those adopted in the present work are found in Appendix V.

p_i, the annual precession for the date T_i of the catalogue to be reduced;

p_0, that for the fundamental epoch T_0;

s, the secular variation.

$\Delta\alpha_i$, the reduction of the catalogue position to the adopted fundamental equinox.

Then, $$\Delta\alpha_i = \tfrac{1}{2}(p_0 + p_i)(T_0 - T_i)$$

or $$\Delta\alpha_i = (T_0 - T_i)p_i + \tfrac{1}{2}(T_0 - T_i)^2 s = [p_i + \tfrac{1}{2}s(T_0 - T_i)][T_0 - T_i].$$

In such a case as this, what is wanted is the position referred to the equinox T_0, as observed at the epoch T_i, without the application of proper motion for the interval. Hence, if the preceding formulae are used, the precession p_0 for the date T_0 should not be computed with the actual place of the star at that epoch, but with the place as it would be found without applying the proper motion from T_i to T_0. But it is only in the case of exceptionally large proper motions that attention to this point is necessary. The criterion is whether the change in the position of the star produced by the proper motion during the interval is large enough to materially affect the precession p_0.

It must also be noted that the formulae cease to be applicable when the star is so near the pole that the angle S for the interval $T_0 - T_i$ cannot be treated as infinitesimal. The trigonometric method should be used in all these exceptional cases.

196. Formation and solution of the equations.

Having the results of the reductions as arranged in the scheme (a), § 193, the problem is to find the position and proper motion which will best satisfy the observations. We may in all but the extremest cases regard the proper motion as constant when referred to the pole of T_0. Even in the exceptional extreme cases, we may proceed on the supposition of uniform proper motion if only we regard the result as the value of a uniformly variable proper motion at the mean epoch of all the observations. Thus we may represent each reduced R.A. (or Dec.) as giving an equation of condition of the form

$$\alpha_0 + \mu t = \text{reduced } \alpha.$$

Here we apply the method developed in § 39 in the following way:

We shall designate the reduced R.A.'s or Decs. by $\alpha_1, \alpha_2, \dots \alpha_n$, it being noted that these symbols now have not the same meaning as in (a), being all reduced to one equinox.

The solution may be effected by arranging the several results in columns in tabular form, as follows:

Column 1: the abbreviated designation of the several star catalogues, in the order of time.

Column 2: the several values of t_i, the mean date of the observing for each catalogue. It will not be necessary to write the century in full, and, in fact, it may be most convenient to write down instead of a year the interval in years before or after 1850. Then, in the case of Bradley's catalogue, the earliest of all, we should have $t = -0{\cdot}95$, and for all dates before 1850 t would be negative. In this column it will ordinarily be most convenient to use the century as the unit, which is done by simply putting a decimal point before the tens of years.

Column 3: the reduced values of the R.A. α_i (or Declination) for the common fundamental epoch T_0. In all ordinary cases it is only necessary to write down the seconds of the coordinate, the hours or degrees and minutes being commonly the same for all catalogues.

Column 4, the assigned weights. Practically a single significant figure will be enough to use for this purpose, or two for the numbers between 10 and 15. That is to say, a weight of 54 may be called 50, one of 56, 60, etc. While it is true that the numerical result may be slightly different, the difference will never be more than a small fraction of the uncertainty. In fact the weight is always under any circumstances a most uncertain datum with which to deal.

These four columns contain our data complete. The next step is to multiply the columns 2 and 3, t and α, by the weights, writing the products in columns 5 and 6. We also form the product wt^2 in each line, which we may do by multiplying wt by t, and which goes into column 7, and $wt\alpha$, which goes into column 8. As a check w may be multiplied mentally by the square of T.

The sum of each column from 4 to 8 will then be formed, giving the values of W, $[t]$, $[x]$, or $[\alpha]$, $[tt]$ and $[tx]$, or $[t\alpha]$.

The value of z from (63) of § 39 will be the concluded seconds of the coordinate for the epoch T_0, and y will be the proper motion for the mean of all the epochs, referred to the pole of T_0. In the exceptional case of large proper motions of stars near the pole, it will be necessary to make a reduction for the changing values of the proper motions even when referred to the same equator and equinox. The theory of this has already been set forth.

197. Use of the central date.

While the preceding method embodies all the operations really necessary for the result, there will be a certain advantage both in symmetry of method and probable freedom from error by adopting the modification developed in § 40. The writer believes that the additional ease and symmetry thus secured will compensate for the slightly increased labour.

By this method we do not form wt^2 column 7, but, after completing column 6, find the mean date of all the observing, or value of t_0 by the equation

$$t_0 = \frac{[wt]}{W}.$$

This mean date may be called the *central date*. It has the property already pointed out, that the most likely value of α for that date is

$$\alpha_0 = \frac{[w\alpha]}{W}$$

independently of the proper motion; and also at this epoch the weight of the position derived from all the observing is a maximum, and diminishes symmetrically with the time before and after this epoch.

Having found the central epoch, we use it as that from which t is counted, following the method of § 40. The columns following (6) will then contain the successive quantities

$$t_i - \frac{[wt]}{W} \equiv \tau\,; \; w\tau^2\,; \; w\tau\alpha.$$

Yet more perspicuity will be secured if, instead of writing $w\tau\alpha$ in column 8, we subtract the weighted mean α_0 of all the α's from each separate α, calling the residual r, and form $w\tau r$. This will make it easy to see the relation between the residuals and proper motions, and in case of any serious divergence or error will bring it out. No other solution of an equation will then be necessary. We shall have for the proper motion

$$\mu = \frac{[\tau r]}{[\tau\tau]}.$$

Since α_0 is the definitive coordinate for the central epoch, the value for the epoch T_0 will be

$$\alpha_0 + \mu(T_0 - \text{Cent. ep.}).$$

The proper motion μ will be that for the central date.

The following form of computation is that above suggested. The Catalogues named in the first column are only taken as examples, and have no preference over others.

1 Catalogue.	2 Mean t	3 Sec. of a	4 Wt. w	5 wt	6 wa	7 wt^2 or τ	8 wta or r	9 $w\tau$	10 $w\tau^2$	11 $w\tau r$
Brad., 1755,	− ·95	a_1	w_1	$w_1 t_1$	$w_1 a_1$	τ_1	r_1	$w_1 \tau_1$	$w_1 \tau_1^2$	$w_1 \tau_1 r_1$
Piazzi, 1800,	− ·50	a_2	w_2	$w_2 t_2$	$w_2 a_2$	τ_2	r_2	$w_2 \tau_2$	$w_2 \tau_2^2$	$w_2 \tau_2 r_2$
Argel., 1830,	− ·20	a_3	w_3	$w_3 t_3$	$w_3 a_3$	τ_3	r_3	$w_3 \tau_3$	$w_3 \tau_3^2$	$w_3 \tau_3 r_3$
Pond, 1830,	− ·20	a_4	w_4	$w_4 t_4$	$w_4 a_4$	τ_4	r_4	$w_4 \tau_4$	$w_4 \tau_4^2$	$w_4 \tau_4 r_4$
Grh., 1840,	− ·10	a_5	w_5	$w_5 t_5$	$w_5 a_5$	τ_5	r_5	$w_5 \tau_5$	$w_5 \tau_5^2$	$w_5 \tau_5 r_5$
Pulk., 1845,	− ·05	a_6	w_6	$w_6 t_6$	$w_6 a_6$	τ_6	r_6	$w_6 \tau_6$	$w_6 \tau_6^2$	$w_6 \tau_6 r_6$
Grh., 1872,	+ ·22	a_7	w_7	$w_7 t_7$	$w_7 a_7$	τ_7	r_7	$w_7 \tau_7$	$w_7 \tau_7^2$	$w_7 \tau_7 r_7$
Pulk., 1885,	+ ·35	a_8	w_8	$w_8 t_8$	$w_8 a_8$	τ_8	r_8	$w_8 \tau_8$	$w_8 \tau_8^2$	$w_8 \tau_8 r_8$
Grh., 1890,	+ ·40	a_9	w_9	$w_9 t_9$	$w_9 a_9$	τ_9	r_9	$w_9 \tau_9$	$w_9 \tau_9^2$	$w_9 \tau_9 r_9$
Mt. Ham., 1900,	+ ·50	a_{10}	w_{10}	$w_{10} t_{10}$	$w_{10} a_{10}$	τ_{10}	r_{10}	$w_{10} \tau_{10}$	$w_{10} \tau_{10}^2$	$w_{10} \tau_{10} r_{10}$
Sums,	—	—	W	$[wt]$	$[wa]$	—	—	0	$[\tau\tau]$	$[\tau r]$

The rigorous application of the above method will be impracticable when a catalogue gives only one coordinate, or when the mean date of observation is materially different for the two coordinates, because in each of these cases we have, strictly speaking, no rigorous determination of a position of the star for a definite epoch, which is necessary for the reduction to T_0. Hence the latter cannot be made except, in the first case, by using an approximate computed value of the missing coordinate, and, in the second case, by reducing the two observed coordinates to the same epoch with an assumed proper motion. In all ordinary cases either of these courses may be taken without leading to error. In exceptional cases it is preferable to adopt the second method.

198. Method of correcting provisional data.

The first step in the application of this method is the preparation of an ephemeris of the mean R.A. and Dec. of the star for the several catalogue dates T_1, T_2, etc. In the case of the standard stars, such an ephemeris may easily be formed from data given in general catalogues of standard stars. Among those which may be used for this purpose are the catalogue of 1098 standard clock and zodiacal stars found in *Astronomical Papers of the American Ephemeris*, vol. i., the general catalogue of standard stars in vol. ix. of the same papers, and the standard catalogues in Publications of the *Astronomischen Gesellschaft*, Nos. xiv. and xvii.

If, from any of these catalogues, we compute positions of a star for the epochs of observation, each excess of an observed over a computed position will give a correction to the latter for the date in question.

In making the comparison, attention must, of course, be paid to the proper motion of the star between the mean date of all the observations used in forming the catalogue place and the date of the equinox to which the place is referred. If no proper motion is applied in the catalogue, it will be necessary to apply the provisional value μ_0. If that actually applied is materially different from the provisional value, the necessary correction should be made.

In any case, the date for which the correction holds good is not that to which the catalogue has been reduced, but the mean date of the observations.

The results of the comparisons will be that at the epochs

$$t_1, t_2, \dots t_n,$$

the provisional place of the star seemed to require the corrections

$$c_1, c_2, \dots c_n,$$

c being, in each case, the excess of the R.A. (or Dec.) of the catalogue over the provisional one.

In nearly all cases it may be assumed that this correction should increase uniformly with the time. We may, therefore, proceed as in method 1, the result being, instead of a mean position and a proper motion, corrections to the assumed mean position and proper motion.

If we compute and apply this correction for any or all the dates for which provisional places have been computed, the result will be the corrected places for the same dates.

199. Special method for close polar stars.

The preceding method presupposes that the angle S at the star is small. When such is not the case, a more rigorous proceeding is necessary, which we shall now briefly indicate.

The elements of position to be corrected are:

$$\alpha_0, \quad \delta_0, \quad \mu_\alpha, \quad \mu_\delta, \qquad \text{(7)}$$

the provisional R.A., Dec., and proper motion of the star at a certain date T_0. A coordinate α or δ at any other date T is a function of these four quantities, so that we may write

$$\left.\begin{aligned} \alpha &= f_1(\alpha_0, \delta_0, \mu_\alpha, \mu_\delta) \\ \delta &= f_2(\alpha_0, \delta_0, \mu_\alpha, \mu_\delta) \end{aligned}\right\} \qquad \text{(8)}$$

If, following the general method found in §§ 34, 35, we apply symbolic corrections $\Delta\alpha_0$, $\Delta\delta_0$, etc., to the elements (7), the effect of these corrections upon the place α will be given by

$$\left.\begin{aligned} \Delta\alpha &= \frac{d\alpha}{d\alpha_0}\Delta\alpha_0 + \frac{d\alpha}{d\delta_0}\Delta\delta_0 + \frac{d\alpha}{d\mu_\alpha}\Delta\mu_\alpha + \frac{d\alpha}{d\mu_\delta}\Delta\mu_\delta \\ \Delta\delta &= \frac{d\delta}{d\alpha_0}\Delta\alpha_0 + \frac{d\delta}{d\delta_0}\Delta\delta_0 + \frac{d\delta}{d\mu_\alpha}\Delta\mu_\alpha + \frac{d\delta}{d\mu_\delta}\Delta\mu_\delta \end{aligned}\right\} \qquad \text{(9)}$$

The rigorous method is to compute these differential coefficients for each of the catalogue dates, and put their numerical values in the right-hand member of (9), while in the left-hand member we put for $\Delta\alpha$ and $\Delta\delta$ the excess of the catalogue or observed over the provisional coordinate. Thus we shall have a conditional equation between the four unknown quantities $\Delta\alpha_0$, $\Delta\delta_0$, $\Delta\mu_\alpha$, and $\Delta\mu_\delta$, and by solving all these equations we derive the values of the corrections to the fundamental elements.

When the angle S is small, we shall have very nearly

$$\left.\begin{aligned} \frac{d\alpha}{d\alpha_0} &= \frac{d\delta}{d\delta_0} = 1 \\ \frac{d\alpha}{d\mu_\alpha} &= \frac{d\delta}{d\mu_\delta} = T_i - T_0 \end{aligned}\right\}, \qquad (10)$$

while the value of the other coefficients will be so small that the terms multiplied by them may be dropped. We should then reproduce the equation, the solution of which is given in the scheme of § 197. When this approximate method is not sufficiently exact, we must seek for the accurate values of the differential coefficients. These we can find from the equations of the form (8), which give the values of α and δ in terms of α_0 and δ_0 through the system of equations

$$\left.\begin{aligned} \mathrm{a}_0 &= \alpha_0 + \mu_\alpha t + \zeta_0 \\ \delta_1 &= \delta_0 + \mu_\delta t \\ \cos\delta \sin \mathrm{a} &= \cos\delta_1 \sin \mathrm{a}_0 \\ \cos\delta\cos \mathrm{a} &= \cos\theta\cos\delta_1\cos \mathrm{a}_0 - \sin\theta\sin\delta_1 \\ \sin\delta &= \sin\theta\cos\delta_1\cos \mathrm{a}_0 + \cos\theta\sin\delta_1 \\ \alpha &= \mathrm{a} + z \end{aligned}\right\} \qquad (11)$$

We see from these equations that

$$\left.\begin{aligned} \frac{d\alpha}{d\alpha_0} &= \frac{d\mathrm{a}}{d\mathrm{a}_0}, \quad \frac{d\alpha}{d\delta_0} = \frac{d\mathrm{a}}{d\delta_1} \\ \frac{d\delta}{d\delta_0} &= \frac{d\delta}{d\delta_1}, \quad \frac{d\delta}{d\alpha_0} = \frac{d\delta}{d\mathrm{a}_0} \end{aligned}\right\} \qquad (12)$$

The quantities which enter into the equation (11) are all parts of the spherical triangle SP_0P formed by the star and the two mean poles. The preceding derivations are expressed in terms of these parts as follows:

$$\left.\begin{aligned}
\frac{d\alpha}{d\alpha_0} &= \frac{\cos\delta_0}{\cos\delta}\cos S \\
\frac{d\alpha}{d\delta_0} &= \frac{\sin S}{\cos\delta} \\
\frac{d\delta}{d\alpha_0} &= -\cos\delta_0 \sin S \\
\frac{d\delta}{d\delta_0} &= \cos S
\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

The values of α and δ in (8) are found by trigonometric reduction by the method of Chapter X. The value of $\Delta\alpha$ and $\Delta\delta$ in (9) are the corrections to this reduced place given by observations. The equations (9) are then solved by the method of least squares for the four unknown quantities which they contain, resulting in corrections to the adopted provisional positions and proper motions.

This method will become more and more necessary as more stars near the pole have to be investigated, and as the period over which observations extend is lengthened. An application of it to the four north polar stars most used will be found in volume viii. of the *Astronomical Papers of the American Ephemeris.*

NOTES AND REFERENCES.

The mass of astronomical literature relating to positions of the fixed stars is so great that it is not possible, in the present connection, to do more than cite the principal independent catalogues of stars, and offer some suggestions as to the literature of the subject. From what has already been said of the history of the subject it will be seen that the determination of positions of the fixed stars by meridian observations has formed a large fraction of the work of the leading observatories since 1750. The instruments, the system of observation, and the methods of reduction and combination have been so frequently imperfect that the question what results are worth using often becomes one of much difficulty, the decision of which must be left to the investigator himself. To this diversity of material must be added lack of continuity in observations and in systematic forms and methods of publication.

Since the middle of the last century, following the example of Greenwich, it has been quite usual for observatories making regular meridian observations to publish in each annual volume the mean positions, for the beginning of the year, of all the stars observed on the meridian during the year. At intervals of a few years these annual positions have been generally, but not always, combined into a single catalogue, reduced to some convenient equinox near the mean of the times of observation. But there are still several series of observations, some of which are probably as good as any made during their time, which, although published, have never been completely reduced. The question whether it would be profitable to utilize such observations is one that frequently arises, but has to be postponed for want of the means necessary to effect the reduction.

Besides the independent volumes issued by observatories, the volumes of the *Astronomische Nachrichten*, the number of which will, before many years, pass the 200 mark, contain a vast amount of material of every kind relating to the subject, which should be accessible to the investigator who wishes to have all the aids which may possibly be useful in his work. Discussions relating to the positions of stars are also found in the *Astronomical Journal*, which has now reached its 25th volume.

Of material contained in these and other serials it will be necessary to cite only that most uniformly essential, namely, the systematic corrections to various catalogues, and the weights to be assigned to the given positions as a function of the number of observations on which each result depends.

Boss's system of corrections is found in *Astronomical Journal*, vol. xxiii., pp. 191-211.

Auwers' system of reductions is found in *Astronomische Abhandlungen als Ergänzungshefte zu den Astronomischen Nachrichten*; Nr. 7, *Tafeln zur Reduction von Sterncatalogen auf das System des Fundamentalcatalogs des Berliner Jahrbuchs.*

Newcomb's system of corrections is found in *Astronomical Papers of the American Ephemeris*, vol. viii., chapter iv.

The differences between the reductions given by these different authors arise not only from the differences of the fundamental systems, but from differences in the principles on which the corrections were derived. The principal difference in principle is that in forming his system Newcomb required more evidence that a systematic difference was necessary than did either Auwers or Boss.

Auwers' tables of weights are found in the *Ast. Nach.*, vol. 151, S. 225-274, under the title : *Gewichstafeln für Sterncataloge.*

The assignment of weights is of necessity largely a matter of judgment, based on what is known of the methods of making the observations and constructing the catalogue. Marked diversity in the different systems is therefore to be expected.

LIST OF INDEPENDENT STAR CATALOGUES.

THE following is a list of the principal independent catalogues of precision which may be available in investigating positions and proper motions of stars. The term "catalogue of precision" is used in a somewhat broad sense, including all catalogues which were intended to be more than simple lists of stars. Many are no doubt admitted which, on critical examination, will be found below others that have been excluded.

From the list are also omitted observations of stars in zones, and catalogues constructed from them. The most important of these are the zones observed by Bessel and by Argelander, well-known catalogues from which have been published by Weisse and by Oeltzen. Annual catalogues are also excluded, whether they have been combined or not.

Another class excluded is that in which the given positions are not independent, but are derived by a combination of other observations than those made especially for the catalogue in question.

In tabulating and comparing the results derived from different catalogues it is necessary to have the briefest distinctive designation of each. This is commonly either the abbreviated name of the observatory, or the name of the author, followed by the date of the catalogue. For the latter is chosen sometimes the equinox of reference and sometimes the mean date of the observations, commonly the former. In the list which follows it is the name of the observatory or place of observation which is generally given. Boss introduced the system of abbreviating the name of the observatory to its first and last letters which is convenient in writing but not always sufficiently explicit. Auwers uses sometimes the name of the observatory and sometimes that of the author of the catalogue.

CATALOGUES MADE AT NORTHERN OBSERVATORIES.

THE first complete reduction and discussion of Bradley's observations was made by Bessel and published in 1818 under the title:

Fundamenta Astronomiae pro anno MDCCLV deducta ex observationibus viri incomparabilis JAMES BRADLEY. *In Specula Astronomica Grenovicensi per Annos* 1750-1762 *Institutis. Auctore* FRIDERICO WILHELMO BESSEL. Regiomonti, 1818.

This work is now superseded by that of Auwers of which the designation and title are:

AUWERS-BRADLEY, 1755.—*Neue Reduction der Bradley'schen Beobachtungen, aus den Jahren* 1750-1762, *von* ARTHUR AUWERS. 3 volumes, St. Petersburg, 1882-1903.

The catalogue is found in the third volume. The first volume contains a valuable discussion and comparison of the observations, which will serve as an excellent model to the astronomical student desiring to perfect himself in methods of discussion. The R.A.'s of Auwers leave no question open which it would be profitable to discuss. But such is not the case with the declinations, his work on which has been examined by the present author as well as by Boss.*

Auwers-Mayer, 1755.—*Tobias Mayer's Sternverzeichniss nach den Beobachtungen auf der Göttinger Sternwarte in den Jahren* 1756-1760. *Neu Bearbeitet* von Arthur Auwers. Leipzig, 1894.

This work is based on observations by Tobias Mayer, made shortly after the epoch of Bradley, with whose work it favourably compares. The number of stars is, however, rather small.

Piazzi, 1880.—*Praecipuarum Stellarum inerrantium Positiones mediae ex observationibus*, 1792-1813. Folio, Panormi, 1814.

This catalogue when constructed was vastly superior to any that preceded it, and is still of value in determining proper motions. But it is now far behind modern requirements. It is being reconstructed by Dr. Herman S. Davis, under the auspices of the Carnegie Institution. Until this work is completed and published it is scarcely worth while to make use of the catalogue except for stars not observed by Bradley.

Groombridge, 1810.—*A Catalogue of Circumpolar Stars, deduced from observations of Stephen Groombridge, Esq. Reduced to January* 1, 1810. Edited by George Biddell Airy, Esq., A.M., *Astronomer Royal.* London, 1838.

The observations on which this catalogue is based were made by an enthusiastic amateur at Blackheath, and are valuable from their early date, and the number of circumpolar stars included. The above cited publication by Airy has been the only one hitherto available, but a re-reduction has recently been completed at the Greenwich Observatory, and the catalogue based upon it is entitled—

New Reduction of Groombridge's Circumpolar Catalogue. By Frank W. Dyson *and* William G. Thackeray *under the direction of* Sir William H. M. Christie. London Admiralty, 1905.

The principal defect in Groombridge's observations is that very few observations were made below the pole, and in consequence the error of his instrument in azimuth cannot be fixed with all desirable certainty.

Pond-Auwers, 1815.—*Mittlere Oerte von* 570 *Sternen . . . aus den unter Direction von Pond*, 1811-1819, *angestellen Beobachtungen.* Von A. Auwers. Berlin Akademie, 1902.

Königsberg, 1820.—*Neue Reduction der Königsberger Declinationen* 1820, von W. Döllen. Found in *Recueil de Mémoires présentés à l'Académie des Sciences par les Astronomes de Poulkova*, vol. i., St. Petersburg, 1853.

This reduction includes only about 60 fundamental stars, to the determination of which Bessel devoted special attention.

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* *Astronomical Papers of the American Ephemeris*, viii., p. 194; *Ast. Jour.*, vol. xxiii.

Dorpat, 1830.—Struve's Fundamental Catalogue for this epoch is found in his *Stellarum Fixarum Positiones Mediae Epocha* 1830. *Auctore* F. G. W. Struve. Petropoli, 1852.

Argelander, 1830.—*DLX Stellarum Fixarum Positiones Mediae Ineunte Anno* 1830. Helsingforsiae, 1835.

This catalogue of 560 stars, almost all of the brighter class, including especially the fundamental stars, is based on Argelander's observations at Abo before his removal to Bonn.

Pond, 1830.—*A Catalogue of* 1,112 *Stars, reduced from observations made at the Royal Observatory at Greenwich, from the years* 1815 *to* 1833. London, 1833.

The observations on which this catalogue is based are probably good when measured by the standard of the period. Their combination in the catalogue is, however, not carried out in the best way, and the re-reduction and recombination of the whole is to be desired. This is partly done by Auwers in the work Pond, 1815, above cited. The result of Chandler's discussion of the standard declinations is found in *Ast. Jour.*, xiv. A correction to the catalogue declinations on account of the refractions is tabulated by Auwers in *Ast. Nach.*, vol. 134, col. 52.

Cambridge, 1830.—*The First Cambridge Catalogue of* 726 *Stars, deduced from the Observations made at the Cambridge Observatory, from* 1828 *to* 1835 ; *reduced to January* 1, 1830, *by* George Biddell Airy, Esq., *Astronomer Royal, etc.*

This work is extracted from *Memoirs of the Royal Astronomical Society*, vol. xi., pages 21 to 45, London, 1840.

The probable errors of this catalogue are larger than would have been anticipated in a work by Airy. It seems probable that a defective system of reduction and combination has detracted from the precision of the results. If so, a re-reduction of the original observations is desirable.

Königsberg, 1835.—*Beobachtungen von Zodiacalsternen am Reichenbachschen Meridiankreise*, in *Astronomische Beobachtungen auf der Königlichen Universitäts-Sternwarte zu Königsberg, von* Dr. Eduard Luther. Band xxxvii., Zweiter Theil. Königsberg, 1886.

Rumker, 1836.—*Mittlere Oerter von* 12000 *Fix-Sternen*, von Carl Rumker. Hamburg, 1852.

Edinburgh, 1840.—This catalogue is based on observations between 1836 and 1845.

Armagh (Robinson), 1840.—*Places of* 5,345 *Stars observed from* 1828 *to* 1854, *by* Rev. T. R. Robinson. Dublin, 1859.

Gilliss, 1840.—*Astronomical Observations made at the Naval Observatory Washington, by* Lieut. J. M. Gilliss, U.S.N. Washington, 1846.

This work contains a catalogue of 1,248 stars, mostly zodiacal and equatorial, observed in connection with moon-culminations between 1838 and 1842. Only the R.A.'s are independent, the Decs. being taken from the B.A. catalogue.

OXFORD, 1845, 1860, and 1890.—*The Radcliffe Catalogues of Stars.*

The old meridian instrument at the Radcliffe Observatory, Oxford, with which these observations were made, was of inferior construction, and, in consequence, its results require systematic corrections, varying rapidly with the position. The introductions to the annual volumes of the Radcliffe observations, recent papers in the *Monthly Notices*, *R.A.S.*, and comparisons in *Ast. Papers of the American Ephemeris*, viii., 166-167, should be consulted.

CARRINGTON, 1855.—*Catalogue of* 3,735 *Circumpolar Stars observed at the Red Hill Observatory*, 1854-56, *by* RICHARD C. CARRINGTON.

The stars of this catalogue are all situated within 9° of the pole. The instrument was probably not of the best; but the catalogue may be regarded as one of precision and, for the region which it covers, the most complete made up to that time.

GREENWICH, 1855 to 1890.—Since 1836, when Airy took charge of the Greenwich Observatory, catalogues based on the observations through periods ranging from six to ten years have appeared as follows:

Epoch of Reference.				Years.
1840	-	Years of observation,	-	1836-1841.
1845	-	,, ,,	-	1842-1847.
1850	-	,, ,,	-	1848-1853.
1860	-	,, ,,	-	1854-1860.
1864	-	,, ,,	-	1861-1867.
1872	-	,, ,,	-	1868-1876.
1880	-	,, ,,	-	1877-1886.
1890	-	,, ,,	-	1887-1896.

POULKOVA, 1845 to 1892.—The Poulkova standard catalogues have appeared in various volumes of the series—*Observations de Pulkowa, publiées par* OTTO STRUVE, *Directeur, etc.*, and *Publications de l'Observatoire Central Nicolas*—and also independently. In some cases a revised edition of the catalogue, which should be used instead of the original, has been issued. The standard catalogues, in some of which the R.A.'s and Decs. are given in separate publications, are found in the following volumes:

For the Epoch 1845 in volumes i. and iv.
,, ,, 1865 ,, xii.
,, ,, 1885 in série ii., vol. i.
,, ,, 1892 ,, ,, vols. viii.-ix.

A corrected list of the standard declinations for 1845 is published as a supplement to volume iv.

Poulkova catalogues, embracing a larger number of stars, are cited in their chronological order.

POULKOVA, 1855.—*Positions moyennes de* 3542 *étoiles déterminées à l'aide du Cercle Méridien de Poulkova dans les années* 1840-1869. *Observations de Poulkova, Vol. VIII.*

In this catalogue an error was made by the computers in applying the correction for errors of graduation of the meridian circle. It happens, however, that, as the corrections in question vary slowly and regularly from one declination to another, and as all declinations of the stars were reduced to the standard of the vertical circle, the final effect of the error upon the positions as given in the catalogue is unimportant. The subject is, however, discussed very fully in a paper by Backlund, found in the *St. Petersburg Memoirs*, series vii., volume xxxvi., St. Petersburg, 1888.

PARIS, 1845, 1860, and 1875.—*Catalogue de l'Observatoire de Paris. Étoiles observées aux instruments meridiens.* 4 volumes, 4to, 1887-1902.

This catalogue is valuable for the great number of faint stars of which it gives modern positions.

YARNALL, 1860.—*Catalogue of Stars observed at the U.S. Naval Observatory during the years* 1845-1877.

Three editions of this catalogue have appeared, the last being thoroughly revised by Professor Edgar Frisby, U.S.N. The work labours under the disadvantage of including two distinct series of observations; the one beginning in 1845 and coming nearly to a stand-still during the years 1850-1860; the other beginning in the year 1861. The condition of the instruments and the method of using them changed so much during this time that the catalogue as a whole may be considered as a combination of two, the results of which require different systematic corrections.

HARVARD, 1865.—*Annals of Harvard College Observatory*, vol. iv.

This catalogue contains R.A.'s of 506 stars, without declinations.

LEIDEN, 1870.—Declinations of 202 Fundamental Stars, *Annalen der Sternwarte in Leiden*, Band ii., p. 125, and *Ast. Nach.*, lxxx., S. 94.

GLASGOW, 1870.—*Catalogue of* 6415 *Stars deduced from observations made at the Glasgow University Observatory.* By ROBERT GRANT. Glasgow, University Press, 1883.

HARVARD, 1875.—*Catalogue of* 1213 *Stars, observed during the years* 1870-1879 *with the Meridian Circle of Harvard College Observatory, by* WILLIAM A. ROGERS; *Harvard Annals*, volume xv., part i.

WASHINGTON, 1875.—*The Second Washington Catalogue of Stars from observations with the transit circle at the U.S. Naval Observatory from* 1866 *to* 1891, *by* PROFESSOR J. R. EASTMANN, U.S.N.*

POULKOVA, 1875.—*Catalog von* 5634 *Sternen für die Epoche* 1875 *aus den Beobachtungen am Pulkowaer Meridiankreise während der Jahre* 1874-1880, *von* H. ROMBERG. *Supplement III. aux Observations de Poulkova*, St. Pétersbourg, 1891.

BERLIN, 1875.—*Ableitung der Rectascensionen der Sterne des Fundamental-*

* It should be noted that the systematic corrections found in the first page of *Ast. Papers of the Am. Eph.*, Vol. VIII., to the declinations of this catalogue are not applicable to the printed declinations, but only to an unpublished original.

Cataloges der Astronomischen Gesellschaft aus den von H. Romberg in den Jahren 1869-1873, *angestellten Beobachtungen von* Dr. A. Marcuse. *Beobachtungs-Ergebnisse der Königlichen Sternwarte zu Berlin.* Heft No. 4. Berlin, 1888.

Ibid.—*Resultate von Beobachtungen von* 521 *Bradley'schen Sternen, am grossen Berliner Meridiankreise, von* Dr. E. Becker. Berlin Observations, 1881.

Armagh, 1875.—*Second Armagh Catalogue of* 3300 *Stars for the epoch* 1875, *by* T. R. Robinson *and* J. L. E. Dreyer.

Astronomische Gesellschaft, 1875.—*Katalog der Astronomischen Gesellschaft, Erste Abtheilung,* +80° bis −2°. In fifteen Parts, of which Part II. (70° to 75°) has not been published. Leipzig, 1890-1902.

Ibid.; *Zweite Abtheilung*:—Pt. II., −6° bis −10°. Leipzig, 1904-, is the only section which has yet appeared.

Oxford, 1890.—*Catalogue of* 6,424 *Stars for the Epoch* 1890, *formed from observations made at the Radcliffe Observatory, Oxford, during the years* 1880 *to* 1893, *by* E. J. Stone.

Berlin, 1890.—*Ergebnisse der* 1886-1891 *am grossen Meridiankreise der Berliner Sternwarte angestellen Beobachtungen der Jahrbuchsterne, von* F. Küstner. *Astronomische Nachrichten*, vol. cxlii., S. 113-134.

Madison, 1890.—*Publications of the Washburn Observatory of the University of Wisconsin*, vol. viii. *Meridian Circle Observations*, 1887-1892. Madison, Wis., 1893.

Glasgow, 1890.—*Second Glasgow Catalogue of* 2156 *Stars from observations made during the years* 1886-1892. By Robert Grant. Glasgow, University Press, 1892.

Munich, 1892.—*Untersuchungen über die astronomische Refraction mit einer Bestimmung der Polhöhe von München und ihrer Schwankungen von November* 1891 *bis October* 1893 *und einem Katalog der absoluten Declinationen von* 116 *Fundamental-Sternen, von* Dr. Julius Bauschinger. München, 1896.

Poulkova, 1895.—*Catalog von* 781 *Zodiacalsternen für Aequinoctium und Epoch* 1895.0, *von* M. Ditschenko *und* J. Segboth. St. Pétersbourg, 1903.

Mount Hamilton, 1895.—*Observations upon selected Stars of the Astronomische Gesellschaft Catalogue made with the meridian circle of the Lick Observatory by* Mr. R. H. Tucker, *during the years* 1894-95. *Ast. Jour.* vol. xvii., No. 408. *Publications of the Lick Observatory*, vol. iv., 302.

Berlin (Battermann), 1895.—*Beobachtungs-Ergebnisse der königlichen Sternwarte zu Berlin.* Heft 8. Berlin, 1899.

Berlin (Battermann), 1900, *Ibid.* Heft 10.

Mount Hamilton, 1900.—*Results of Observations of Circumpolar Stars, Zodiacal Stars, and Southern Stars of Piazzi. Publications of Lick Observatory*, vol. vi.

Auwers applies large systematic corrections to the declinations of the southern stars in this catalogue, which are probably necessary on account of the usual tables of refraction not being correct for an altitude of 1300 metres above sea level.

CINCINNATI, 1890, 1895, and 1900.—*Publications of the Cincinnati Observatory*, Nos. 13, 14, and 15, by JERMAIN G. PORTER, Director. These catalogues give observed positions of 2000, 2030, and 4280 stars respectively.

CATALOGUES FROM TROPICAL AND SOUTHERN OBSERVATORIES.

LACAILLE, 1750.—*A Catalogue of* 9766 *Stars in the Southern Hemisphere from the observations of the Abbé de Lacaille made at the Cape of Good Hope, in the years* 1751-1752. By FRANCIS BAILY, ESQ. London, 1847.

The origin of this catalogue is mentioned in the preceding chapter. From its very nature it cannot be regarded as a catalogue of precision, but it is cited because its positions may be useful in the case of stars not found in other catalogues.

PARAMATTA, 1825.—*Catalogue of* 7385 *Stars, chiefly in the southern hemisphere, from the observations made in* 1822-26 *at the Observatory at Paramatta, New South Wales, founded by* SIR THOMAS MACDOUGALL BRISBANE; *the Catalogue by* MR. WILLIAM RICHARDSON. London, 1835.

The observations on which this catalogue was based were made with the transit instrument and mural circle; a few of them by Sir Thomas Brisbane himself, but mostly by Mr. Charles Rümker, later of Hamburg, and Mr. Dunlop. The work is of importance as being the first catalogue of precision embracing stars too far south to be visible in Europe. So far as the writer is aware, the precision of the results has never been tested by modern methods.

FALLOWS, 1830.—*A Catalogue of* 425 *Stars observed during the years* 1829-31 *at the Cape Observatory, reduced and published by* G. B. AIRY. *Memoirs of the Royal Astronomical Society*, vol. xix.

ST. HELENA (JOHNSON), 1830.—*A Catalogue of* 606 *Principal Fixed Stars in the Southern Hemisphere, deduced from observations at the Observatory, St. Helena, from November*, 1829, *to April*, 1833, *by* MANUEL J. JOHNSON. London, 1835.

CAPE (HENDERSON), 1833.—*Thos. Henderson on the Declinations of the Principal Fixed Stars, deduced from observations made at the Observatory, Cape of Good Hope, in the years* 1832 *and* 1833. *Memoirs of the Royal Astronomical Society*, vol. x.

MADRAS (TAYLOR), 1835.—*Taylor's General Catalogue of Stars from observations made at the Madras Observatory during the years* 1831-1842. *Revised and edited by* A. M. W. DOWNING, ESQ. Edinburgh, 1901.

This catalogue, based on the work of an industrious observer, is of decided value, but suffers from the imperfection of the instruments with which the observations were made. The R.A.'s especially are affected by a systematic error varying with the declination of the star, which probably arose from an error in the collimation of the transit.

SANTIAGO, 1850.—*A Catalogue of* 1963 *Stars ...... together with a Catalogue of* 290 *Double Stars, the whole from observations made at Santiago, Chili, during the years* 1850-51-52, *by the U.S. Naval Astronomical Expedition to the Southern Hemisphere*, LIEUT. JAMES M. GILLISS, LL.D., *Superintendent. Washington Observations for* 1868, Appendix I., Washington, 1870.

SANTIAGO, 1855.—*Catalogo de Ascenciones Rectas i Distancias Polares medias ...... deducidas de las observaciones en los años* 1853, 1854, *i* 1855. Santiago de Chile, 1859.

SANTIAGO, 1860.—*Ascenciones Rectas i Distancias Polares de las estrellas observadas en los años de* 1856 *á* 1860 *con el Circulo Meridiano. Observatorio Nacional*, Santiago de Chile, 1875.

MELBOURNE, 1870.—*First Melbourne General Catalogue of* 1,227 *Stars for the Epoch* 1870, *deduced from observations extending from* 1863 *to* 1870, *made at the Melbourne Observatory.* Melbourne, 1874.

CORDOBA, 1875.—The Argentine General Catalogue. *Resultados del Observatorio Nacional Argentino*, vol. xiv., Cordoba, 1886.

MADRAS, 1875.—*Results of Observations of the Fixed Stars made with the Madras Meridian Circle*, vol. ix. *General Catalogue.* Madras, 1899.

CAPE, 1840-1900.—Catalogues constructed from the observations at the Cape have appeared for the epochs :

1840	From observations	1834-40.
1850	„ „	1849-52.

Also, *Cape Catalogue of* 1159 *Stars, Royal Observatory, Cape of Good Hope,* 1856 *to* 1861, *reduced to the epoch* 1860 *by* E. J. STONE, 1873.

Catalogue of 1905 *Stars for the Equinox* 1865, *from observations made at the Royal Observatory, Cape of Good Hope*, 1861 *to* 1870, *by* SIR THOMAS MACLEAR. *Reduced by* DAVID GILL. 8vo, London, 1899.

Catalogue of 12,441 *Stars for the Epoch* 1880, *from observations made at the Royal Observatory, Cape of Good Hope*, 1871-1879, *by* E. J. STONE. 4to, London, 1881.

Catalogue of 1713 *Stars for the Equinox* 1885, *from observations made at the Royal Observatory, Cape of Good Hope*, 1879-1885. DAVID GILL. 4to, London, 1894.

Catalogue of 3007 *Stars for the Equinox* 1890, *from observations made at the Royal Observatory, Cape of Good Hope*, 1885-1895. DAVID GILL. 4to, London, 1898.

APPENDIX OF FORMULAE AND TABLES.

THE tables found in this appendix are, for the most part, sufficiently explained in connection with them. The following are cases in which the explanation is either wanting, or not given with all desirable fulness:

Appendix II., Table IA., gives the day of the Julian period corresponding to the initial day of each century, the reckoning being according to the Julian calendar up to 1599, and the Gregorian from 1600 on.

Table IB. gives the reduction to the beginning of each year, and IC. to the zero date of each month in the year. The Julian day for any date is therefore found by adding to the sum of the numbers from the three tables the excess of the day of the month above the next preceding date found in Table IC., being careful to enter the right column of the latter, according as the year is or is not bissextile.

In the case of the dates B.C., the century only is taken as negative, and Table IA. must then be entered with the century next larger numerically than the given century. It must also be noted that, in this case, the astronomical reckoning of the years to which the table corresponds is numerically less by 1 than the usual reckoning of the chronologists. (See 66.)

As an example of this case, let the day of the Julian period be desired corresponding to the chronological date B.C. 306, January 18. The astronomical year is -305, and the date $-400+95$, Jan. 18. The computation of the Julian date is as follows:

Table IA., century, - - -	1 574 958
„ IB., year, - - -	34 698
„ IC., Jan. 10+8, - -	18
Julian day required, - -	1 609 674

The fact that the years 1600, 2000, and 2400 are bissextile, while the numbers IA. are arranged for centuries which begin with a common year, makes it necessary to diminish the day for these years by 1, using -1 instead of 0 for the year.

Tables II. and III. will be readily understood by a study of Chap V., §§ 61 and 65.

Table IV.—Each column of hundredths of a day in this table is followed by a column containing the equivalent in hours, minutes, and seconds. The column next following is one hundredth of this, and therefore gives the equivalent for the third and fourth decimals of the day. The third column following gives the equivalent for the fifth and sixth decimals. As an example the reduction of 0·720 853 to h., m., and s. is

$$17 \text{ h. } 16 \text{ m. } 48 \text{ s.} + 1 \text{ m. } 9{\cdot}12 \text{ s.} + 4{\cdot}58 \text{ s.} = 17 \text{ h. } 18 \text{ m. } 1{\cdot}70 \text{ s.}$$

For the reverse reduction, we find in the second column of either part of the table the h., m., and s. next smaller than the given ones, and write down the corresponding two figures of the argument. Then we take mentally the excess of the given h., m., and s. above that of the table, enter the third column for the next two decimals, and so on.

Table V. gives in the second column of each part the time of beginning of each solar year during the twentieth century. During the first 72 years of the century the moment of beginning is always after Greenwich mean noon of the zero date of the year, that is December 31 of the year preceding. Beginning with the year 1973, the solar bissextile years begin before January 0 as thus defined, and the date is therefore negative in the table. These data being especially useful in tables and ephemerides of apparent places of stars, the other arguments necessary for computing these places are given in Tables V., VI., and VII. for sidereal days reckoned from the beginning of the solar year.

As to the form of these tables, it should be noted that the "tabular year," frequently used in astronomical tables, begins on January 0 of common years as here, but on January 1 of leap years. During the months of January and February the days of this tabular year are less by 1 than of the civil year. But, in the present tables, this tabular year is not used; hence for dates after March 1 the day of the year will be 1 greater in leap years than in common years.

APPENDIX III.

Centennial values of the precessional motions.—These motions are computed as shown in Chapter IX., Section 1. The precessions given in most catalogues of stars are annual, and the secular variation of each is its change for 100 years, and is therefore $\frac{1}{100}$ the value of $D_T^2\alpha$ or $D_T^2\delta$. It may be computed with the same coefficients, using the annual instead of the centennial motions, except as to the small terms factored by $\mu_\alpha\mu_\delta$ and μ_α^2.

If the secular variation of the precession alone is required, μ_α and μ_δ should be added to p_α and p_δ instead of $2\mu_\alpha$ and $2\mu_\delta$; and if the effect of proper motions is to be entirely omitted, as in reducing the geometric place unchanged, μ_α and μ_δ should be taken as zero.

APPENDIX IV.

The development of the methods set forth in this appendix is fully given in Chapter X., Section II., where examples of the use of the tables will be found.

APPENDIX V.

In most star-catalogues between 1850 and 1900 the precessions are those of Struve-Peters. They may be reduced to the new values by applying -38 to the number from the first column of Table XVI. to obtain Δp_α, and multiplying the number from the second column of XVI. by nat. $\tan\delta$ from XVII., which will give Δp_δ. If the precessions are annual, the units of the correction will then be $0^s{\cdot}00001$ and $0''{\cdot}0001$, respectively; if centennial, $0^s{\cdot}001$ and $0''{\cdot}01$.

APPENDIX VI.

These tables are used for the rapid conversion of ecliptic into equatorial coordinates and *vice versa*, where a greater accuracy than to a coarse fraction of a minute is not required.

Table XX. is arranged for the rapid conversion of small corrections in one set of coordinates to corrections in the other. As this conversion is rarely necessary, except in the case of the moon and planets, it is given only between the limits -5° and $+5^\circ$ of latitude.

APPENDIX VII.

The condensed table of refraction here given is only approximate. Refractions correct to $\pm 0''\cdot 1$ may, however, be found from it when the zenith distance is not too great, and the deviation of the temperature and pressure from the adopted standard not too wide.

APPENDIX IX.

Three-place tables of logarithms and trigonometric functions are given, because they are not usually at hand, and should be used in all cases when sufficiently accurate. It is often easier to form a product of three figures by three with numbers than by logarithms especially if a table of products is used. The natural values of the trigonometric are therefore often convenient to use instead of their logarithms. But the latter are preferable in forming a product of more than two factors.

APPENDIX I.

CONSTANTS AND FORMULAE IN FREQUENT USE.

A.—Constants with their Logarithms.

	Numbers.	Logarithms.
Ratio of circumference to diameter	$\pi = 3{\cdot}141\ 592\ 65$	0·497 149 9
	$2\pi = 6{\cdot}283\ 185\ 31$	0·798 179 9
	$\pi^2 = 9{\cdot}869\ 604\ 40$	0·994 299 7
	$\sqrt{\pi} = 1{\cdot}772\ 453\ 85$	0·248 574 9
Degrees in circumference - -	360	2·556 302 5
Minutes ,, - -	21 600	4·334 453 8
Seconds ,, - -	1 296 000	6·112 605 0
Degrees in radian - - - -	$57^\circ{\cdot}295\ 779\ 5$	1·758 122 6
Minutes ,, - - -	$3\ 437'{\cdot}746\ 77$	3·536 273 9
Seconds ,, - - -	$206\ 264''{\cdot}806$	5·314 425 1
Seconds of time in radian -	$13\ 750^s{\cdot}987$	4·138 333 9
Length of arc of one degree - -	0·017 453 29	8·241 877 4 – 10
,, ,, ,, minute - -	0·000 290 89	6·463 726 1 – 10
,, ,, ,, second - -	0·000 004 848	4·685 574 9 – 10
,, ,, ,, second of time	0·000 072 722	5·861 666 1 – 10
Hours in one day - - - -	24	1·380 211 2
Minutes ,, - - - -	1 440	3·158 362 5
Seconds ,, - - - -	86 400	4·936 513 7
Days in Julian Year - - -	365·25	2·562 590 2
Hours ,, - - -	8 766	3·942 801 5
Minutes ,, - - -	525 960	5·720 952 7
Seconds ,, -	31 557 600	7·499 104 0
Days in Solar Year - - -	365·2422	2·562 580 9
Hours ,, - - -	8 765·813	3·942 792 2
Minutes ,, - - -	525 948·77	5·720 943 4
Seconds ,, - -	31 556 926·0	7·499 094 7

The following values of the constants for reduction of places of the fixed stars were most in use between 1830 and 1900, but are now being superseded by the values adopted in the present work.

The values of the variable quantities are given for 1850, from which epoch T is reckoned in centuries.

Annual general precession, Bessel	- -	$50''\cdot2357 + \cdot0244\ T$
,, ,, ,, Peters-Struve		$50\ \cdot2524 + \cdot0227\ T$
,, motion of pole, Bessel	- -	$20\ \cdot0547 - \cdot0097\ T$
,, ,, ,, Peters-Struve	-	$20\ \cdot0564 - \cdot0086\ T$
Constant of nutation, Bessel	- - -	$8''\cdot977$
,, ,, Peters-Struve	-	$9\ \cdot223$
Constant of aberration, Bessel	- -	$20''\cdot255$
,, ,, Struve	- -	$20\ \cdot445$
,, ,, Nyrén	- -	$20\ \cdot492$
,, ,, Newcomb	- -	$20\ \cdot50$
,, ,, Chandler	- -	$20\ \cdot53$

Dimensions of the geoid according to the leading authorities. Helmert's b here given is his latest result.

	a, metres.	b, metres.	Mean Radius. Metres.	Compression.
Helmert	6 378 000	6 356 612	6 370 843	$1 \div 298\cdot20$
Clarke	6 378 249	6 356 515	6 370 997	$1 \div 293\cdot46$
Bessel	6 377 397	6 356 079	6 370 282	$1 \div 299\cdot15$

B.—Formulae for the Solution of Spherical Triangles.

$a,\quad b,\quad c$ the sides.

$A,\quad B,\quad C$ the opposite angles.

CASE I.—Given two sides a, b and the included angle C.

$$\sin c \sin A = \sin a \sin C,$$
$$\sin c \cos A = \cos a \sin b - \sin a \cos b \cos C,$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

If we compute k and K from

$$k \sin K = \sin a \cos C,$$
$$k \cos K = \cos a,$$

then

$$\sin c \cos A = k \sin (b - K),$$
$$\cos c = k \cos (b - K).$$

The Gaussian equations for this case, not advantageous unless A, B, and c are all required, are

$$\sin \tfrac{1}{2}c \sin \tfrac{1}{2}(A - B) = \cos \tfrac{1}{2}C \sin \tfrac{1}{2}(a - b),$$
$$\sin \tfrac{1}{2}c \cos \tfrac{1}{2}(A - B) = \sin \tfrac{1}{2}C \sin \tfrac{1}{2}(a + b),$$
$$\cos \tfrac{1}{2}c \sin \tfrac{1}{2}(A + B) = \cos \tfrac{1}{2}C \cos \tfrac{1}{2}(a - b),$$
$$\cos \tfrac{1}{2}c \cos \tfrac{1}{2}(A + B) = \sin \tfrac{1}{2}C \cos \tfrac{1}{2}(a + b).$$

CASE II.—Given two angles and the intermediate side A, B, c.

$$\sin C \sin a = \sin A \sin c,$$
$$\sin C \cos a = \cos A \sin B + \sin A \cos B \cos c,$$
$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

If we compute h and H from

$$h \sin H = \cos A,$$
$$h \cos H = \sin A \cos c,$$

then

$$\sin C \cos a = h \cos (B - H),$$
$$\cos C = h \sin (B - H).$$

The Gaussian equations for this case are formed by writing those of Case I. in the order 2, 4, 1, 3, interchanging the two members of each equation.

CASE III.—Given the three sides.

$$s = \tfrac{1}{2}(a + b + c),$$
$$m^2 = \frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s},$$
$$\tan \tfrac{1}{2}A = \frac{m}{\sin (s - a)},$$
$$\tan \tfrac{1}{2}B = \frac{m}{\sin (s - b)},$$
$$\tan \tfrac{1}{2}C = \frac{m}{\sin (s - c)}.$$

CASE IV.—Given the three angles.

$$S = \tfrac{1}{2}(A + B + C),$$
$$M^2 = \frac{-\cos S}{\cos (S - A) \cos (S - B) \cos (S - C)},$$
$$\tan \tfrac{1}{2}a = M \cos (S - A),$$
$$\tan \tfrac{1}{2}b = M \cos (S - B),$$
$$\tan \tfrac{1}{2}c = M \cos (S - C).$$

CASE V.—Given two sides and the angle opposite one of them, a, b, A.

$$\sin B = \frac{\sin A \sin b}{\sin a} \quad \text{(two values of } B\text{)},$$
$$\tan \tfrac{1}{2}C = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)} \cot \tfrac{1}{2}(A + B),$$
$$\tan \tfrac{1}{2}c = \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} \tan \tfrac{1}{2}(a + b).$$

CASE VI.—Given two angles and the side opposite one of them, A, B, a.

$$\sin b = \frac{\sin a \sin B}{\sin A} \quad \text{(two values of } b\text{)},$$

$$\tan \tfrac{1}{2}c = \frac{\cos \frac{1}{2}(A+B)}{\cos \frac{1}{2}(A-B)} \tan \tfrac{1}{2}(a+b),$$

$$\tan \tfrac{1}{2}C = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \cot \tfrac{1}{2}(A+B).$$

In right spherical triangles the fundamental equations take the following forms:

c the hypothenuse.

$$\begin{aligned} \sin c \sin A &= \sin a, \\ \sin c \cos A &= \cos a \sin b, \\ \cos c &= \cos a \cos b, \\ \sin b &= \sin B \sin c, \\ \sin a \cos b &= \cos B \sin c, \\ \tan a &= \cos B \tan c, \\ \tan a &= \tan A \sin b. \end{aligned}$$

C.—Differentials of the Parts of a Spherical Triangle.

The partial derivatives of any part of a spherical triangle with respect to the three other parts on which it depends are found from that one of the following equations which contains the differentials of the four variable parts. (Comp. § 6.)

$$\begin{aligned} -\sin C da + \cos b \sin A dc + \sin b dA + \cos C \sin a dB &= 0, \\ -\sin A db + \cos c \sin B da + \sin c dB + \cos A \sin b dC &= 0, \\ -\sin B dc + \cos a \sin C db + \sin a dC + \cos B \sin c dA &= 0. \end{aligned}$$

$$\begin{aligned} -\sin b \sin C da + dA + \cos c dB + \cos b dC &= 0, \\ -\sin c \sin A db + dB + \cos a dC + \cos c dA &= 0, \\ -\sin a \sin B dc + dC + \cos b dA + \cos a dB &= 0. \end{aligned}$$

$$\begin{aligned} -da + \cos C db + \cos B dc + \sin c \sin B dA &= 0, \\ -db + \cos A dc + \cos C da + \sin a \sin C dB &= 0, \\ -dc + \cos B da + \cos A db + \sin b \sin A dC &= 0. \end{aligned}$$

The remaining forms may be written by leaving any one pair of letters, say a and A, unaltered and interchanging the other two, say B with C and b with c.

APPENDIX II.

To find the Day of the Julian Period corresponding to any Day of the Julian Calendar to 1600, *or of the Gregorian Calendar after* 1600.

TABLE Ia. FOR CENTURY.

−1900	1 027 083
−1800	1 063 608
−1700	1 100 133
−1600	1 136 658
−1500	1 173 183
−1400	1 209 708
−1300	1 246 233
−1200	1 282 758
−1100	1 319 283
−1000	1 355 808
−900	1 392 333
−800	1 428 858
−700	1 465 383
−600	1 501 908
−500	1 538 433
−400	1 574 958
−300	1 611 483
−200	1 648 008
−100	1 684 533
0	1 721 058
+100	1 757 583
200	1 794 108
300	1 830 633
400	1 867 158
500	1 903 683
600	1 940 208
700	1 976 733
800	2 013 258
900	2 049 783
1000	2 086 308
1100	2 122 833
1200	2 159 358
1300	2 195 883
1400	2 232 408
1500	2 268 933
1600	2 305 448
1700	2 341 972
1800	2 378 496
1900	2 415 020
2000	2 451 545
2100	2 488 069
2200	2 524 593
2300	2 561 117
2400	2 597 642

TABLE Ib. FOR YEAR IN CENTURY.

00	0 *or* −1	50	18 262
01	365	51	18 627
02	730	52	18 992
03	1 095	53	19 358
04	1 460	54	19 723
05	1 826	55	20 088
06	2 191	56	20 453
07	2 556	57	20 819
08	2 921	58	21 184
09	3 287	59	21 549
10	3 652	60	21 914
11	4 017	61	22 280
12	4 382	62	22 645
13	4 748	63	23 010
14	5 113	64	23 375
15	5 478	65	23 741
16	5 843	66	24 106
17	6 209	67	24 471
18	6 574	68	24 836
19	6 939	69	25 202
20	7 304	70	25 567
21	7 670	71	25 932
22	8 035	72	26 297
23	8 400	73	26 663
24	8 765	74	27 028
25	9 131	75	27 393
26	9 496	76	27 758
27	9 861	77	28 124
28	10 226	78	28 489
29	10 592	79	28 854
30	10 957	80	29 219
31	11 322	81	29 585
32	11 687	82	29 950
33	12 053	83	30 315
34	12 418	84	30 680
35	12 783	85	31 046
36	13 148	86	31 411
37	13 514	87	31 776
38	13 879	88	32 141
39	14 244	89	32 507
40	14 609	90	32 872
41	14 975	91	33 237
42	15 340	92	33 602
43	15 705	93	33 968
44	16 070	94	34 333
45	16 436	95	34 698
46	16 801	96	35 063
47	17 166	97	35 429
48	17 531	98	35 794
49	17 897	99	36 159

TABLE Ic. FOR THE DAY.

		Common Year.	Leap Year.
Jan.	0	0	0
	10	10	10
	20	20	20
	30	30	30
Feb.	0	31	31
	10	41	41
	20	51	51
Mar.	0	59	60
	10	69	70
	20	79	80
	30	89	90
Apr.	0	90	91
	10	100	101
	20	110	111
May	0	120	121
	10	130	131
	20	140	141
	30	150	151
June	0	151	152
	10	161	162
	20	171	172
July	0	181	182
	10	191	192
	20	201	202
	30	211	212
Aug.	0	212	213
	10	222	223
	20	232	233
	30	242	243
Sept.	0	243	244
	10	253	254
	20	263	264
Oct.	0	273	274
	10	283	284
	20	293	294
	30	303	304
Nov.	0	304	305
	10	314	315
	20	324	325
Dec.	0	334	335
	10	344	345
	20	354	355
	30	364	365

TABLE IIa.

To Convert Mean into Sidereal Time.

Mean Time.		Correction. +		Mean Time.		Correction. +		Mean Time.		Correction. +	
h.	m.	m.	s.	h.	m.	m.	s.	h.	m.	m.	s.
0	0	0	0·00	8	0	1	18·85	16	0	2	37·70
	10		1·64		10		20·50		10		39·35
	20		3·29		20		22·14		20		40·99
	30		4·93		30		23·78		30		42·63
	40		6·57		40		25·42		40		44·28
	50		8·21		50		27·07		50		45·92
1	0	0	9·86	9	0	1	28·71	17	0	2	47·56
	10		11·50		10		30·35		10		49·20
	20		13·14		20		31·99		20		50·85
	30		14·78		30		33·64		30		52·49
	40		16·43		40		35·28		40		54·13
	50		18·07		50		36·92		50		55·77
2	0	0	19·71	10	0	1	38·56	18	0	2	57·42
	10		21·36		10		40·21		10		59·06
	20		23·00		20		41·85		20	3	0·70
	30		24·64		30		43·49		30		2·34
	40		26·28		40		45·14		40		3·99
	50		27·93		50		46·78		50		5·63
3	0	0	29·57	11	0	1	48·42	19	0	3	7·27
	10		31·21		10		50·06		10		8·92
	20		32·86		20		51·71		20		10·56
	30		34·50		30		53·35		30		12·20
	40		36·14		40		54·99		40		13·84
	50		37·78		50		56·64		50		15·49
4	0	0	39·43	12	0	1	58·28	20	0	3	17·13
	10		41·07		10		59·92		10		18·77
	20		42·71		20	2	1·56		20		20·42
	30		44·35		30		3·21		30		22·06
	40		46·00		40		4·85		40		23·70
	50		47·64		50		6·49		50		25·34
5	0	0	49·28	13	0	2	8·13	21	0	3	26·99
	10		50·92		10		9·78		10		28·63
	20		52·57		20		11·42		20		30·27
	30		54·21		30		13·06		30		31·91
	40		55·85		40		14·70		40		33·56
	50		57·50		50		16·35		50		35·20
6	0	0	59·14	14	0	2	17·99	22	0	3	36·84
	10	1	0·78		10		19·63		10		38·48
	20		2·42		20		21·28		20		40·13
	30		4·07		30		22·92		30		41·77
	40		5·71		40		24·56		40		43·41
	50		7·35		50		26·20		50		45·06
7	0	1	9·00	15	0	2	27·85	23	0	3	46·70
	10		10·64		10		29·49		10		48·34
	20		12·28		20		31·13		20		49·98
	30		13·92		30		32·77		30		51·63
	40		15·57		40		34·42		40		53·27
	50		17·21		50		36·06		50		54·91

Correction for Minutes and Seconds.		
m.	s.	s.
0	10	0·03
	20	0·05
	30	0·08
	40	0·11
	50	0·14
1	0	0·16
	10	0·19
	20	0·22
	30	0·25
	40	0·27
	50	0·30
2	0	0·33
	10	0·36
	20	0·38
	30	0·41
	40	0·44
	50	0·47
3	0	0·49
	10	0·52
	20	0·55
	30	0·57
	40	0·60
	50	0·63
4	0	0·66
	10	0·68
	20	0·71
	30	0·74
	40	0·77
	50	0·79
5	0	0·82
	10	0·85
	20	0·88
	30	0·90
	40	0·93
	50	0·96
6	0	0·99
	10	1·01
	20	1·04
	30	1·07
	40	1·10
	50	1·12
7	0	1·15
	10	1·18
	20	1·21
	30	1·23
	40	1·26
	50	1·29
8	0	1·31
	10	1·34
	20	1·37
	30	1·40
	40	1·42
	50	1·45
9	0	1·48
	10	1·50
	20	1·53
	30	1·56
	40	1·59
	50	1·62

TABLE IIB.

To Convert Sidereal into Mean Time.

Sidereal Time.		Correction. –		Sidereal Time.		Correction. –		Sidereal Time.		Correction. –	
h.	m.	m.	s.	h.	m.	m.	s.	h.	m.	m.	s.
0	0	0	0·00	8	0	1	18·64	16	0	2	37·27
	10		1·64		10		20·28		10		38·91
	20		3·28		20		21·91		20		40·55
	30		4·92		30		23·55		30		42·19
	40		6·55		40		25·19		40		43·83
	50		8·19		50		26·83		50		45·46
1	0	0	9·83	9	0	1	28·47	17	0	2	47·10
	10		11·47		10		30·10		10		48·74
	20		13·11		20		31·74		20		50·38
	30		14·74		30		33·38		30		52·02
	40		16·38		40		35·02		40		53·66
	50		18·02		50		36·66		50		55·29
2	0	0	19·66	10	0	1	38·30	18	0	2	56·93
	10		21·30		10		39·93		10		58·57
	20		22·94		20		41·57		20	3	0·21
	30		24·57		30		43·21		30		1·85
	40		26·21		40		44·85		40		3·48
	50		27·85		50		46·49		50		5·12
3	0	0	29·49	11	0	1	48·12	19	0	3	6·76
	10		31·13		10		49·76		10		8·40
	20		32·76		20		51·40		20		10·04
	30		34·40		30		53·04		30		11·68
	40		36·04		40		54·68		40		13·32
	50		37·68		50		56·32		50		14·95
4	0	0	39·32	12	0	1	57·96	20	0	3	16·59
	10		40·96		10		59·59		10		18·23
	20		42·60		20	2	1·23		20		19·87
	30		44·23		30		2·87		30		21·51
	40		45·87		40		4·51		40		23·14
	50		47·51		50		6·15		50		24·78
5	0	0	49·15	13	0	2	7·78	21	0	3	26·42
	10		50·79		10		9·42		10		28·06
	20		52·42		20		11·06		20		29·70
	30		54·06		30		12·70		30		31·34
	40		55·70		40		14·34		40		32·97
	50		57·34		50		15·98		50		34·61
6	0	0	58·98	14	0	2	17·61	22	0	3	36·25
	10	1	0·62		10		19·25		10		37·89
	20		2·25		20		20·89		20		39·53
	30		3·89		30		22·53		30		41·16
	40		5·53		40		24·17		40		42·80
	50		7·17		50		25·80		50		44·44
7	0	1	8·81	15	0	2	27·44	23	0	3	46·08
	10		10·44		10		29·08		10		47·72
	20		12·08		20		30·72		20		49·36
	30		13·72		30		32·36		30		51·00
	40		15·36		40		34·00		40		52·63
	50		17·00		50		35·64		50		54·27

Correction for Minutes and Seconds.		
m.	s.	s.
0	10	0·03
	20	0·05
	30	0·08
	40	0·11
	50	0·14
1	0	0·16
	10	0·19
	20	0·22
	30	0·25
	40	0·27
	50	0·30
2	0	0·33
	10	0·35
	20	0·38
	30	0·41
	40	0·44
	50	0·47
3	0	0·49
	10	0·52
	20	0·55
	30	0·57
	40	0·60
	50	0·63
4	0	0·66
	10	0·68
	20	0·71
	30	0·74
	40	0·76
	50	0·79
5	0	0·82
	10	0·85
	20	0·87
	30	0·90
	40	0·93
	50	0·96
6	0	0·98
	10	1·01
	20	1·04
	30	1·06
	40	1·09
	50	1·12
7	0	1·15
	10	1·17
	20	1·20
	30	1·23
	40	1·26
	50	1·28
8	0	1·31
	10	1·34
	20	1·37
	30	1·39
	40	1·42
	50	1·45
	0	1·47
	10	1·50
	20	1·53
	30	1·56
	40	1·58
	50	1·61

TABLE III.

Time into Arc and vice versa.

h. m.	°	h. m.	°	h. m.	°	h. m.	°	h. m.	°	h. m.	°
0 0	0	3 0	45	6 0	90	9 0	135	12 0	180	15 0	225
4	1	4	46	4	91	4	136	4	181	4	226
8	2	8	47	8	92	8	137	8	182	8	227
12	3	12	48	12	93	12	138	12	183	12	228
16	4	16	49	16	94	16	139	16	184	16	229
0 20	5	3 20	50	6 20	95	9 20	140	12 20	185	15 20	230
24	6	24	51	24	96	24	141	24	186	24	231
28	7	28	52	28	97	28	142	28	187	28	232
32	8	32	53	32	98	32	143	32	188	32	233
36	9	36	54	36	99	36	144	36	189	36	234
0 40	10	3 40	55	6 40	100	9 40	145	12 40	190	15 40	235
44	11	44	56	44	101	44	146	44	191	44	236
48	12	48	57	48	102	48	147	48	192	48	237
52	13	52	58	52	103	52	148	52	193	52	238
56	14	56	59	56	104	56	149	56	194	56	239
1 0	15	4 0	60	7 0	105	10 0	150	13 0	195	16 0	240
4	16	4	61	4	106	4	151	4	196	4	241
8	17	8	62	8	107	8	152	8	197	8	242
12	18	12	63	12	108	12	153	12	198	12	243
16	19	16	64	16	109	16	154	16	199	16	244
1 20	20	4 20	65	7 20	110	10 20	155	13 20	200	16 20	245
24	21	24	66	24	111	24	156	24	201	24	246
28	22	28	67	28	112	28	157	28	202	28	247
32	23	32	68	32	113	32	158	32	203	32	248
36	24	36	69	36	114	36	159	36	204	36	249
1 40	25	4 40	70	7 40	115	10 40	160	13 40	205	16 40	250
44	26	44	71	44	116	44	161	44	206	44	251
48	27	48	72	48	117	48	162	48	207	48	252
52	28	52	73	52	118	52	163	52	208	52	253
56	29	56	74	56	119	56	164	56	209	56	254
2 0	30	5 0	75	8 0	120	11 0	165	14 0	210	17 0	255
4	31	4	76	4	121	4	166	4	211	4	256
8	32	8	77	8	122	8	167	8	212	8	257
12	33	12	78	12	123	12	168	12	213	12	258
16	34	16	79	16	124	16	169	16	214	16	259
2 20	35	5 20	80	8 20	125	11 20	170	14 20	215	17 20	260
24	36	24	81	24	126	24	171	24	216	24	261
28	37	28	82	28	127	28	172	28	217	28	262
32	38	32	83	32	128	32	173	32	218	32	263
36	39	36	84	36	129	36	174	36	219	36	264
2 40	40	5 40	85	8 40	130	11 40	175	14 40	220	17 40	265
44	41	44	86	44	131	44	176	44	221	44	266
48	42	48	87	48	132	48	177	48	222	48	267
52	43	52	88	52	133	52	178	52	223	52	268
56	44	56	89	56	134	56	179	56	224	56	269
3 0	45	6 0	90	9 0	135	12 0	180	15 0	225	18 0	270

TABLE III.—*Concluded.*

Time into Arc and vice versa.

h. m.	°	h. m.	°	m. s.	′	m. s.	′	s.	″	s.	″
18 0	270	21 0	315	0 0	0	2 0	30	0·000	0	2·000	30
4	271	4	316	4	1	4	31	0·066	1	2·066	31
8	272	8	317	8	2	8	32	0·133	2	2·133	32
12	273	12	318	12	3	12	33	0·200	3	2·200	33
16	274	16	319	16	4	16	34	0·266	4	2·266	34
18 20	275	21 20	320	0 20	5	2 20	35	0·333	5	2·333	35
24	276	24	321	24	6	24	36	0·400	6	2·400	36
28	277	28	322	28	7	28	37	0·466	7	2·466	37
32	278	32	323	32	8	32	38	0·533	8	2·533	38
36	279	36	324	36	9	36	39	0·600	9	2·600	39
18 40	280	21 40	325	0 40	10	2 40	40	0·666	10	2·666	40
44	281	44	326	44	11	44	41	0·733	11	2·733	41
48	282	48	327	48	12	48	42	0·800	12	2·800	42
52	283	52	328	52	13	52	43	0·866	13	2·866	43
56	284	56	329	56	14	56	44	0·933	14	2·933	44
19 0	285	22 0	330	1 0	15	3 0	45	1·000	15	3·000	45
4	286	4	331	4	16	4	46	1·066	16	3·066	46
8	287	8	332	8	17	8	47	1·133	17	3·133	47
12	288	12	333	12	18	12	48	1·200	18	3·200	48
16	289	16	334	16	19	16	49	1·266	19	3·266	49
19 20	290	22 20	335	1 20	20	3 20	50	1·333	20	3·333	50
24	291	24	336	24	21	24	51	1·400	21	3·400	51
28	292	28	337	28	22	28	52	1·466	22	3·466	52
32	293	32	338	32	23	32	53	1·533	23	3·533	53
36	294	36	339	36	24	36	54	1·600	24	3·600	54
19 40	295	22 40	340	1 40	25	3 40	55	1·666	25	3·666	55
44	296	44	341	44	26	44	56	1·733	26	3·733	56
48	297	48	342	48	27	48	57	1·800	27	3·800	57
52	298	52	343	52	28	52	58	1·866	28	3·866	58
56	299	56	344	56	29	56	59	1·933	29	3·933	59
20 0	300	23 0	345	2 0	30	4 0	60	2·000	30	4·000	60
4	301	4	346								
8	302	8	347								
12	303	12	348								
16	304	16	349								
20 20	305	23 20	350								
24	306	24	351								
28	307	28	352								
32	308	32	353								
36	309	36	354								
20 40	310	23 40	355								
44	311	44	356								
48	312	48	357								
52	313	52	358								
56	314	56	359								
21 0	315	24 0	360								

TABLE IV.

To change Decimals of a Day to Hours, Minutes, and Seconds, and vice versa.

d.	h. m. s.	m. s.	s.	d.	h. m. s.	m. s.	s.
0·01	0 14 24	0 8·64	0·09	0·51	12 14 24	7 20·64	4·41
0·02	0 28 48	0 17·28	0·17	0·52	12 28 48	7 29·28	4·49
0·03	0 43 12	0 25·92	0·26	0·53	12 43 12	7 37·92	4·58
0·04	0 57 36	0 34·56	0·35	0·54	12 57 36	7 46·56	4·67
0·05	1 12 0	0 43·20	0·43	0·55	13 12 0	7 55·20	4·75
0·06	1 26 24	0 51·84	0·52	0·56	13 26 24	8 3·84	4·84
0·07	1 40 48	1 0·48	0·60	0·57	13 40 48	8 12·48	4·92
0·08	1 55 12	1 9·12	0·69	0·58	13 55 12	8 21·12	5·01
0·09	2 9 36	1 17·76	0·78	0·59	14 9 36	8 29·76	5·10
0·10	2 24 0	1 26·40	0·86	0·60	14 24 0	8 38·40	5·18
0·11	2 38 24	1 35·04	0·95	0·61	14 38 24	8 47·04	5·27
0·12	2 52 48	1 43·68	1·04	0·62	14 52 48	8 55·68	5·36
0·13	3 7 12	1 52·32	1·12	0·63	15 7 12	9 4·32	5·44
0·14	3 21 36	2 0·96	1·21	0·64	15 21 36	9 12·96	5·53
0·15	3 36 0	2 9·60	1·30	0·65	15 36 0	9 21·60	5·62
0·16	3 50 24	2 18·24	1·38	0·66	15 50 24	9 30·24	5·70
0·17	4 4 48	2 26·88	1·47	0·67	16 4 48	9 38·88	5·79
0·18	4 19 12	2 35·52	1·56	0·68	16 19 12	9 47·52	5·88
0·19	4 33 36	2 44·16	1·64	0·69	16 33 36	9 56·16	5·96
0·20	4 48 0	2 52·80	1·73	0·70	16 48 0	10 4·80	6·05
0·21	5 2 24	3 1·44	1·81	0·71	17 2 24	10 13·44	6·13
0·22	5 16 48	3 10·08	1·90	0·72	17 16 48	10 22·08	6·22
0·23	5 31 12	3 18·72	1·99	0·73	17 31 12	10 30·72	6·31
0·24	5 45 36	3 27·36	2·07	0·74	17 45 36	10 39·36	6·39
0·25	6 0 0	3 36·00	2·16	0·75	18 0 0	10 48·00	6·48
0·26	6 14 24	3 44·64	2·25	0·76	18 14 24	10 56·64	6·57
0·27	6 28 48	3 53·28	2·33	0·77	18 28 48	11 5·28	6·65
0·28	6 43 12	4 1·92	2·42	0·78	18 43 12	11 13·92	6·74
0·29	6 57 36	4 10·56	2·51	0·79	18 57 36	11 22·56	6·83
0·30	7 12 0	4 19·20	2·59	0·80	19 12 0	11 31·20	6·91
0·31	7 26 24	4 27·84	2·68	0·81	19 26 24	11 39·84	7·00
0·32	7 40 48	4 36·48	2·76	0·82	19 40 48	11 48·48	7·08
0·33	7 55 12	4 45·12	2·85	0·83	19 55 12	11 57·12	7·17
0·34	8 9 36	4 53·76	2·94	0·84	20 9 36	12 5·76	7·26
0·35	8 24 0	5 2·40	3·02	0·85	20 24 0	12 14·40	7·34
0·36	8 38 24	5 11·04	3·11	0·86	20 38 24	12 23·04	7·43
0·37	8 52 48	5 19·68	3·20	0·87	20 52 48	12 31·68	7·52
0·38	9 7 12	5 28·32	3·28	0·88	21 7 12	12 40·32	7·60
0·39	9 21 36	5 36·96	3·37	0·89	21 21 36	12 48·96	7·69
0·40	9 36 0	5 45·60	3·46	0·90	21 36 0	12 57·60	7·78
0·41	9 50 24	5 54·24	3·54	0·91	21 50 24	13 6·24	7·86
0·42	10 4 48	6 2·88	3·63	0·92	22 4 48	13 14·88	7·95
0·43	10 19 12	6 11·52	3·72	0·93	22 19 12	13 23·52	8·04
0·44	10 33 36	6 20·16	3·80	0·94	22 33 36	13 32·16	8·12
0·45	10 48 0	6 28·80	3·89	0·95	22 48 0	13 40·80	8·21
0·46	11 2 24	6 37·44	3·97	0·96	23 2 24	13 49·44	8·29
0·47	11 16 48	6 46·08	4·06	0·97	23 16 48	13 58·08	8·38
0·48	11 31 12	6 54·72	4·15	0·98	23 31 12	14 6·72	8·47
0·49	11 45 36	7 3·36	4·23	0·99	23 45 36	14 15·36	8·55
0·50	12 0 0	7 12·00	4·32	1·00	24 0 0	14 24·00	8·64

TABLE V.

Greenwich Mean Time of the Beginning of the Adopted Solar Year from 1900 to 2000.—Mean Longitude of the Moon's Node and Perigee.—Moon's Mean Longitude.

Year.	Beginning of Solar Year.	Long. ☾'s Node. ☊	Long. ☾'s Perigee. Π	☾'s Mean Longitude.	Year.	Beginning of Solar Year.	Long. ☾'s Node. ☊	Long. ☾'s Perigee. Π	☾'s Mean Longitude.
	Jan.	°	°	°		Jan.	°	°	°
1900	0·313	259·16	334·4	274·6	1950	0·423	12·11	208·8	63·4
01	0·556	239·82	15·0	47·1	51	0·666	352·77	249·5	196·0
02	0·798	220·48	55·7	179·7	1952B	0·908	333·43	290·2	328·5
03	1·040	201·14	96·4	312·3	53	0·150	314·09	330·9	101·1
1904B	1·282	181·80	137·1	84·9	54	0·392	294·75	11·6	233·7
05	0·524	162·46	177·8	217·5	55	0·634	275·41	52·3	6·3
06	0·767	143·12	218·5	350·0	1956B	0·877	256·07	93·0	138·8
07	1·009	123·78	259·2	122·6	57	0·119	236·73	133·7	271·4
1908B	1·251	104·44	299·9	255·2	58	0·361	217·38	174·4	44·0
09	0·493	85·09	340·6	27·8	59	0·603	198·04	215·0	176·6
10	0·735	65·75	21·3	160·3	1960B	0·845	178·70	255·7	309·1
11	0·978	46·41	61·9	292·9	61	0·088	159·36	296·4	81·7
1912B	1·220	27·07	102·6	65·5	62	0·330	140·02	337·1	214·3
13	0·462	7·73	143·3	198·1	63	0·572	120·68	17·8	346·9
14	0·704	348·39	184·0	330·6	1964B	0·814	101·34	58·5	119·4
15	0·946	329·05	224·7	103·2	65	0·056	82·00	99·2	252·0
1916B	1·189	309·71	265·4	235·8	66	0·299	62·66	139·9	24·6
17	0·431	290·37	306·1	8·4	67	0·541	43·32	180·6	157·2
18	0·673	271·03	346·8	140·9	1968B	0·783	23·97	221·3	289·8
19	0·915	251·68	27·5	273·5	69	0·025	4·63	261·9	62·3
1920B	1·157	232·34	68·2	46·1	70	0·267	345·29	302·6	194·9
21	0·400	213·00	108·8	178·7	71	0·510	325·95	343·3	327·5
22	0·642	193·66	149·5	311·2	1972B	0·752	306·61	24·0	100·1
23	0·884	174·32	190·2	83·8	73	−0·006	287·27	64·7	232·6
1924B	1·126	154·98	230·9	216·4	74	0·236	267·93	105·4	5·2
25	0·368	135·64	271·6	349·0	75	0·478	248·59	146·1	137·8
26	0·611	116·30	312·3	121·6	1976B	0·720	229·25	186·8	270·4
27	0·853	96·96	353·0	254·1	77	−0·037	209·91	227·5	42·9
1928B	1·095	77·62	33·7	26·7	78	0·205	190·56	268·2	175·5
29	0·337	58·27	74·4	159·3	79	0·447	171·22	308·8	308·1
30	0·579	38·93	115·0	291·9	1980B	0·689	151·88	349·5	80·7
31	0·822	19·59	155·7	64·4	81	−0·069	132·54	30·2	213·2
1932B	1·064	0·25	196·4	197·0	82	0·174	113·20	70·9	345·8
33	0·306	340·91	237·1	329·6	83	0·416	93·86	111·6	118·4
34	0·548	321·57	277·8	102·2	1984B	0·658	74·52	152·3	251·0
35	0·790	302·23	318·5	234·7	85	−0·100	55·18	193·0	23·5
1936B	1·033	282·89	359·2	7·3	86	0·142	35·84	233·7	156·1
37	0·275	263·55	39·9	139·9	87	0·385	16·50	274·4	288·7
38	0·517	244·21	80·6	272·5	1988B	0·627	357·15	315·0	61·3
39	0·759	224·86	121·3	45·0	89	−0·131	337·81	355·7	193·9
1940B	1·001	205·52	161·9	177·6	90	0·111	318·47	36·4	326·4
41	0·244	186·18	202·6	310·2	91	0·353	299·13	77·1	99·0
42	0·486	166·84	243·3	82·8	1992B	0·596	279·79	117·8	231·6
43	0·728	147·50	284·0	215·3	93	−0·162	260·45	158·5	4·2
1944B	0·970	128·16	324·7	347·9	94	0·080	241·11	199·2	136·7
45	0·212	108·82	5·4	120·5	95	0·322	221·77	239·9	269·3
46	0·455	89·48	46·1	253·1	1996B	0·564	202·43	280·6	41·9
47	0·697	70·14	86·8	25·7	97	−0·193	183·09	321·3	174·5
48B	0·939	50·80	127·5	158·2	98	0·049	163·74	1·9	307·0
49	0·181	31·45	168·2	290·8	99	0·291	144·40	42·6	79·6
1950	0·423	12·11	208·8	63·4	2000B	0·533	125·06	83·3	212·2

TABLE VI.

Motions of Moon's Node, Perigee, and Mean Longitude.

Sidereal Days.	Solar Date.		Motion of		
			☊	π	☾
			°	°	°
0	Jan.	0·0	− 0·000	+ 0·00	+ 0·00
10		10·0	0·528	1·11	131·40
20		19·9	1·056	2·22	262·81
30		29·9	1·584	3·33	34·21
40	Feb.	8·9	2·112	4·44	165·62
50		18·9	2·640	5·55	297·02
60		28·8	3·169	6·67	68·43
70	Mar.	10·8*	3·697	7·78	199·83
80		20·8*	4·225	8·89	331·24
90		30·8*	4·753	10·00	102·64
100	Apr.	9·7*	5·281	11·11	234·04
110		19·7*	5·809	12·22	5·45
120		29·7*	6·337	13·33	136·85
130	May	9·6*	6·865	14·44	268·26
140		19·6*	7·393	15·55	39·66
150		29·6*	7·921	16·66	171·07
160	June	8·6*	8·450	17·78	302·47
170		18·5*	8·978	18·89	73·88
180		28·5*	9·506	20·00	205·28
190	July	8·5*	10·034	21·11	336·68
200		18·5*	10·562	22·22	108·09
210		28·4*	11·090	23·33	239·49
220	Aug.	7·4*	11·618	24·44	10·90
230		17·4*	12·146	25·55	142·30
240		27·3*	12·674	26·66	273·71
250	Sept.	6·3*	13·202	27·77	45·11
260		16·3*	13·730	28·89	176·52
270		26·3*	14·259	30·00	307·92
280	Oct.	6·2*	14·787	31·11	79·33
290		16·2*	15·315	32·22	210·73
300		26·2*	15·843	33·33	342·13
310	Nov.	5·2*	16·371	34·44	113·54
320		15·1*	16·899	35·55	244·94
330		25·1*	17·427	36·66	16·35
340	Dec.	5·1*	17·955	37·77	147·75
350		15·0*	18·483	38·88	279·16
360		25·0*	19·011	40·00	50·56
370		35·0*	− 19·540	+ 41·11	+ 181·97

* In Bissextile years, the dates after March 1 are to be diminished one day.

TABLE VII.

Motion of Moon's Mean Longitude for Tenths of a Day.

Days (Sidereal).	Motion of ☾	Days (Sidereal).	Motion of ☾
	°		°
0·0	0·00	5·0	65·70
0·1	1·31	5·1	67·02
0·2	2·63	5·2	68·33
0·3	3·94	5·3	69·64
0·4	5·26	5·4	70·96
0·5	6·57	5·5	72·27
0·6	7·88	5·6	73·59
0·7	9·20	5·7	74·90
0·8	10·51	5·8	76·21
0·9	11·83	5·9	77·53
1·0	13·14	6·0	78·84
1·1	14·45	6·1	80·16
1·2	15·77	6·2	81·47
1·3	17·08	6·3	82·79
1·4	18·40	6·4	84·10
1·5	19·71	6·5	85·41
1·6	21·02	6·6	86·73
1·7	22·34	6·7	88·04
1·8	23·65	6·8	89·36
1·9	24·97	6·9	90·67
2·0	26·28	7·0	91·98
2·1	27·60	7·1	93·30
2·2	28·91	7·2	94·61
2·3	30·22	7·3	95·93
2·4	31·54	7·4	97·24
2·5	32·85	7·5	98·55
2·6	34·17	7·6	99·87
2·7	35·48	7·7	101·18
2·8	36·79	7·8	102·50
2·9	38·11	7·9	103·81
3·0	39·42	8·0	105·12
3·1	40·74	8·1	106·44
3·2	42·05	8·2	107·75
3·3	43·36	8·3	109·07
3·4	44·68	8·4	110·38
3·5	45·99	8·5	111·69
3·6	47·31	8·6	113·01
3·7	48·62	8·7	114·32
3·8	49·93	8·8	115·64
3·9	51·25	8·9	116·95
4·0	52·56	9·0	118·26
4·1	53·88	9·1	119·58
4·2	55·19	9·2	120·89
4·3	56·50	9·3	122·21
4·4	57·82	9·4	123·52
4·5	59·13	9·5	124·83
4·6	60·45	9·6	126·15
4·7	61·76	9·7	127·46
4·8	63·07	9·8	128·78
4·9	64·39	9·9	130·09
5·0	65·70	10·0	131·04

APPENDIX III.

TABLE VIII.

Centennial rates of the precessional motions from 1750 to 2000.

Date.	Motion in R.A.	Polar Motion.			
	m_c	n_c	log n_c	n_c	log n_c
	s.	s.		″	
1750	306·955	133·731	2·126 232	2005·96	3·302 323
1775	307·001	133·717	2·126 186	2005·75	3·302 277
1800	307·048	133·703	2·126 140	2005·54	3·302 231
1825	307·094	133·689	2·126 094	2005·32	3·302 185
1850	307·141	133·674	2·126 048	2005·11	3·302 139
1875	307·187	133·660	2·126 001	2004·90	3·302 092
1900	307·234	133·646	2·125 955	2004·68	3·302 046
1925	307·280	133·632	2·125 909	2004·47	3·302 000
1950	307·327	133·617	2·125 863	2004·26	3·301 954
1975	307·373	133·603	2·125 817	2004·04	3·301 908
2000	307·420	133·589	2·125 771	2003·83	3·301 862

Date.	Luni-solar Precession.	General Precession.	Precession from Motion of Ecliptic.	
			in R.A., λ′	in Long., λ′ cos ε
	p	l		
1750	5036″·34	5022″·30	15″·30	14″·03
1775	5036 ·47	5022 ·86	14 ·83	13 ·60
1800	5036 ·59	5023 ·41	14 ·36	13 ·17
1825	5036 ·71	5023 ·97	13 ·89	12 ·74
1850	5036 ·84	5024 ·53	13 ·42	12 ·31
1875	5036 ·96	5025 ·08	12 ·95	11 ·88
1900	5037 ·08	5025 ·64	12 ·48	11 ·45
1925	5037 ·21	5026 ·19	12 ·00	11 ·02
1950	5037 ·33	5026 ·75	11 ·53	10 ·58
1975	5037 ·45	5027 ·31	11 ·06	10 ·15
2000	5037 ·58	5027 ·86	10 ·59	9 ·72

Formulae for the annual precessions.

In R.A., $p_\alpha = m + n \sin\alpha \tan\delta$,

In Dec., $p_\delta = n \cos\alpha$,

where $m = m_c \div 100$; $n = n_c \div 100$.

Formulae for the centennial precessions.

(Same as for the annual precessions, using m_c for m and n_c for n.)

Centennial variations of the centennial variations of α *and* δ.

$$D_T^2\alpha = C_\alpha + A(p_{\alpha c} + 2\mu_\alpha)\cos\alpha$$
$$+ B(p_{\delta c} + 2\mu_\delta)\sin\alpha$$
$$+ [4{\cdot}9866 - 10]\mu_\alpha\mu_\delta \tan\delta,$$
$$D_T^2\delta = C_\delta - [9{\cdot}1637 - 10](p_{\alpha c} + 2\mu_\alpha)\sin\alpha$$
$$- [6{\cdot}7367 - 10]\,\mu_\alpha^2 \sin 2\delta,$$

where μ_α is the centennial proper motion of α in seconds of time, μ_δ that of δ in seconds of arc, and A, B, C_α and C_δ are to be taken from the following tables. (Compare § 146.)

TABLES FOR COMPUTING THE SECULAR VARIATIONS OF THE CENTENNIAL PROPER MOTIONS OF THE STARS.

TABLE IX.

Dec.		Log. A.	Log. B.	Dec.		Log. A.	Log. B.	Dec.		Log. A.	Log. B.
°	′			°	′			°	′		
0	0	....	6·8115	7	0	7·0767	6·8180	14	0	7·3844	6·8377
	10	5·4513	·8115		10	·0871	·8183		10	·3897	·8383
	20	5·7524	·8115		20	·0972	·8186		20	·3950	·8390
	30	5·9285	·8115		30	·1070	·8190		30	·4003	·8396
	40	6·0534	·8116		40	·1167	·8193		40	·4054	·8403
	50	·1503	·8116		50	·1261	·8196		50	·4106	·8409
1	0	6·2295	6·8116	8	0	7·1354	6·8200	15	0	7·4157	6·8416
	10	·2965	·8117		10	·1445	·8204		10	·4207	·8423
	20	·3545	·8117		20	·1534	·8207		20	·4257	·8429
	30	·4057	·8118		30	·1621	·8211		30	·4306	·8437
	40	·4514	·8119		40	·1707	·8215		40	·4355	·8444
	50	·4929	·8119		50	·1791	·8219		50	·4403	·8451
2	0	6·5307	6·8120	9	0	7·1873	6·8223	16	0	7·4451	6·8458
	10	·5655	·8121		10	·1954	·8227		10	·4498	·8465
	20	·5977	·8122		20	·2034	·8231		20	·4545	·8473
	30	·6277	·8123		30	·2112	·8235		30	·4592	·8480
	40	·6558	·8124		40	·2189	·8239		40	·4638	·8488
	50	·6821	·8126		50	·2265	·8244		50	·4684	·8495
3	0	6·7070	6·8127	10	0	7·2339	6·8248	17	0	7·4729	6·8503
	10	·7305	·8128		10	·2412	·8252		10	·4774	·8511
	20	·7528	·8130		20	·2485	·8257		20	·4819	·8519
	30	·7741	·8131		30	·2556	·8262		30	·4863	·8527
	40	·7943	·8133		40	·2626	·8266		40	·4907	·8535
	50	·8137	·8134		50	·2695	·8271		50	·4951	·8543
4	0	6·8322	6·8136	11	0	7·2763	6·8276	18	0	7·4994	6·8551
	10	·8500	·8138		10	·2829	·8281		10	·5037	·8559
	20	·8671	·8140		20	·2896	·8286		20	·5079	·8567
	30	·8836	·8142		30	·2961	·8291		30	·5121	·8576
	40	·8994	·8144		40	·3025	·8296		40	·5163	·8584
	50	·9148	·8146		50	·3088	·8302		50	·5205	·8593
5	0	6·9296	6·8148	12	0	7·3151	6·8307	19	0	7·5246	6·8602
	10	·9439	·8150		10	·3212	·8312		10	·5287	·8610
	20	·9577	·8153		20	·3273	·8318		20	·5327	·8619
	30	·9712	·8155		30	·3334	·8323		30	·5367	·8628
	40	·9842	·8158		40	·3393	·8329		40	·5407	·8637
	50	6·9969	·8160		50	·3452	·8335		50	·5447	·8646
6	0	7·0092	6·8163	13	0	7·3510	6·8341	20	0	7·5487	6·8655
	10	·0212	·8165		10	·3567	·8346		10	·5526	·8665
	20	·0329	·8168		20	·3624	·8352		20	·5565	·8674
	30	·0443	·8171		30	·3680	·8358		30	·5603	·8683
	40	·0554	·8174		40	·3735	·8364		40	·5642	·8693
	50	·0662	·8177		50	·3790	·8371		50	·5680	·8703
7	0	7·0767	6·8180	14	0	7·3844	6·8377	21	0	7·5718	6·8712

TABLE IX.—*Continued.*

Dec.		Log. A.	Log. B.	Dec.		Log. A.	Log. B.	Dec.		Log. A.	Log. B.
°	′			°	′			°	′		
21	0	7·5718	6·8712	29	0	7·7314	6·9279	37	0	7·8647	7·0068
	10	·5755	·8722		10	·7343	·9293		10	·8673	·0087
	20	·5793	·8732		20	·7373	·9307		20	·8700	·0106
	30	·5830	·8741		30	·7402	·9321		30	·8726	·0126
	40	·5867	·8751		40	·7432	·9335		40	·8752	·0145
	50	·5904	·8762		50	·7461	·9350		50	·8778	·0165
22	0	7·5940	6·8772	30	0	7·7490	6·9364	38	0	7·8804	7·0184
	10	·5976	·8782		10	·7520	·9379		10	·8830	·0204
	20	·6012	·8792		20	·7549	·9394		20	8856	·0224
	30	·6048	·8803		30	·7577	·9409		30	·8882	·0244
	40	·6084	·8813		40	·7606	·9424		40	·8908	·0264
	50	·6119	·8824		50	·7635	·9439		50	·8934	·0285
23	0	7·6155	6·8834	31	0	7·7664	6·9454	39	0	7·8960	7·0305
	10	·6190	·8845		10	·7692	9469		10	·8986	·0325
	20	·6224	·8856		20	·7721	·9484		20	·9011	0346
	30	·6259	·8867		30	·7749	·9500		30	·9037	·0367
	40	·6293	·8878		40	·7778	·9515		40	·9063	·0388
	50	·6328	·8889		50	·7806	·9531		50	·9088	·0409
24	0	7·6362	6·8900	32	0	7·7834	6·9547	40	0	7·9114	7·0430
	10	·6396	·8912		10	·7862	·9562		10	·9140	·0451
	20	·6429	·8923		20	·7890	·9578		20	·9165	·0473
	30	·6463	·8935		30	·7918	·9594		30	·9191	·0494
	40	·6496	·8946		40	·7946	·9611		40	·9217	·0516
	50	·6530	·8958		50	·7973	·9627		50	·9242	·0538
25	0	7·6563	6·8969	33	0	7·8001	6·9643	41	0	7·9268	7·0559
	10	·6596	·8981		10	·8029	·9660		10	·9293	·0581
	20	·6628	·8993		20	·8056	·9676		20	·9319	·0604
	30	·6661	·9005		30	·8084	·9693		30	·9344	·0626
	40	·6693	·9017		40	·8111	·9710		40	·9370	·0648
	50	·6726	·9030		50	·8139	·9727		50	·9395	·0671
26	0	7·6758	6·9042	34	0	7·8166	6·9744	42	0	7·9420	7·0694
	10	·6790	·9054		10	·8193	·9761		10	·9446	·0716
	20	·6822	·9067		20	·8220	·9778		20	·9471	·0739
	30	·6853	·9079		30	·8247	·9795		30	·9497	·0762
	40	·6885	·9092		40	·8274	·9813		40	·9522	·0786
	50	·6916	·9105		50	·8301	·9830		50	·9547	·0809
27	0	7·6948	6·9117	35	0	7·8328	6·9848	43	0	7·9573	7·0832
	10	·6979	·9130		10	·8355	·9865		10	·9598	·0856
	20	·7010	·9143		20	·8382	·9883		20	·9623	·0880
	30	·7041	·9156		30	·8409	·9901		30	·9648	·0904
	40	·7072	·9170		40	·8435	·9919		40	·9674	·0928
	50	·7102	·9183		50	·8462	·9938		50	·9699	·0952
28	0	7·7133	6·9196	36	0	7·8489	6·9956	44	0	7·9724	7·0976
	10	·7163	·9210		10	·8515	·9974		10	·9750	·1001
	20	·7193	·9223		20	·8542	6·9993		20	·9775	·1025
	30	·7224	·9237		30	·8568	7·0011		30	·9800	·1050
	40	·7254	·9251		40	·8594	·0030		40	·9825	·1075
	50	·7284	·9265		50	·8621	·0049		50	·9851	·1100
29	0	7·7314	6·9279	37	0	7·8647	7·0068	45	0	7·9876	7·1125

TABLE IX.—*Continued.*

Dec.	Log. A.	Log. B.	Dec.	Log. A.	Log. B.	Dec.	Log. A.	Log. B.
° ′			° ′			° ′		
45 0	7·9876	7·1125	53 0	8·1105	7·2526	61 0	8·2438	7·4404
10	·9901	·1151	10	·1131	·2559	10	·2468	·4449
20	·9927	·1176	20	·1158	·2593	20	·2498	·4495
30	·9952	·1202	30	·1184	·2627	30	·2528	·4542
40	7·9977	·1228	40	·1210	·2661	40	·2559	·4588
50	8·0002	·1253	50	·1237	·2696	50	·2589	·4635
46 0	8·0028	7·1280	54 0	8·1263	7·2731	62 0	8·2619	7·4683
10	·0053	·1306	10	·1290	·2766	10	·2650	·4731
20	·0078	·1332	20	·1317	·2801	20	·2680	·4779
30	·0104	·1359	30	·1343	·2836	30	·2711	·4827
40	·0129	·1385	40	·1370	·2871	40	·2742	·4876
50	·0154	·1412	50	·1397	·2907	50	·2773	·4925
47 0	8·0179	7·1439	55 0	8·1424	7·2943	63 0	8·2804	7·4974
10	·0205	·1467	10	·1451	·2979	10	·2836	·5024
20	·0230	·1494	20	·1478	·3016	20	·2867	·5074
30	·0255	·1521	30	·1505	·3052	30	·2899	·5124
40	·0281	·1549	40	·1532	·3089	40	·2930	·5175
50	·0306	·1577	50	·1559	·3126	50	·2962	·5227
48 0	8·0332	7·1605	56 0	8·1586	7·3164	64 0	8·2994	7·5278
10	·0357	·1633	10	·1613	·3201	10	·3026	·5330
20	·0382	·1661	20	·1641	·3239	20	·3059	·5383
30	·0408	·1689	30	·1668	·3277	30	·3091	·5435
40	·0433	·1718	40	·1696	·3316	40	·3124	·5488
50	·0459	·1747	50	·1723	·3354	50	·3156	·5542
49 0	8·0484	7·1776	57 0	8·1751	7·3393	65 0	8·3189	7·5596
10	·0510	·1805	10	·1779	·3432	10	·3222	·5650
20	·0535	·1835	20	·1806	·3471	20	·3256	·5705
30	·0561	·1864	30	·1834	·3511	30	·3289	·5760
40	·0587	·1894	40	·1862	·3550	40	·3323	·5816
50	·0612	·1923	50	·1890	·3591	50	·3356	·5872
50 0	8·0638	7·1954	58 0	8·1918	7·3631	66 0	8·3390	7·5929
10	·0664	·1984	10	·1946	·3671	10	·3424	·5986
20	·0689	·2014	20	·1974	·3712	20	·3459	·6043
30	·0715	·2045	30	·2003	·3753	30	·3493	·6101
40	·0741	·2076	40	·2031	·3795	40	·3528	·6159
50	·0766	·2106	50	·2060	·3836	50	·3562	·6218
51 0	8·0792	7·2138	59 0	8·2088	7·3878	67 0	8·3597	7·6277
10	·0818	·2169	10	·2117	·3920	10	·3633	·6337
20	·0844	·2200	20	·2146	·3963	20	·3668	·6397
30	·0870	·2232	30	·2175	·4006	30	·3704	·6458
40	·0896	·2264	40	·2203	·4049	40	·3740	·6519
50	·0922	·2296	50	·2232	·4092	50	·3776	·6581
52 0	8·0948	7·2328	60 0	8·2262	7·4136	68 0	8·3812	7·6643
10	·0974	·2361	10	·2291	·4180	10	·3848	·6706
20	·1000	·2393	20	·2320	·4224	20	·3885	·6770
30	·1026	·2426	30	·2350	·4268	30	·3922	·6833
40	·1052	·2459	40	·2379	·4313	40	·3959	·6898
50	·1079	·2492	50	·2409	·4358	50	·3997	·6963
53 0	8·1105	7·2526	61 0	8·2438	7·4404	69 0	8·4034	7·7028

TABLE IX.—*Concluded.*

Dec. °	′	Log. A.	Log. B.	Dec. °	′	Log. A.	Log. B.	Dec. °	′	Log. A.	Log. B.
69	0	8·4034	7·7028	76	0	8·5908	8·0441	83	0	8·8985	8·6397
	10	·4072	·7094		10	·5962	·0543		10	·9090	·6605
	20	·4110	·7161		20	·6017	·0647		20	·9198	·6819
	30	·4149	·7228		30	·6072	·0751		30	·9309	·7038
	40	·4187	·7296		40	·6128	·0857		40	·9423	·7263
	50	·4226	·7365		50	·6185	·0965		50	·9540	·7493
70	0	8·4265	7·7434	77	0	8·6242	8·1073	84	0	8·9660	8·7730
	10	·4305	·7504		10	·6300	·1183		10	·9783	·7974
	20	·4345	·7574		20	·6359	·1295		20	8·9910	·8225
	30	·4385	·7645		30	·6418	·1408		30	9·0040	·8484
	40	·4425	·7717		40	·6479	·1523		40	·0175	·8750
	50	·4465	·7789		50	·6540	·1639		50	·0313	·9025
71	0	8·4506	7·7862	78	0	8·6601	8·1757	85	0	9·0456	8·9309
	10	·4547	·7936		10	·6664	·1877		10	·0604	·9603
	20	·4589	·8010		20	·6727	·1999		20	·0758	8·9907
	30	·4631	·8085		30	·6791	·2122		30	·0916	9·0222
	40	·4673	·8161		40	·6856	·2247		40	·1081	·0549
	50	·4715	·8238		50	·6923	·2374		50	·1252	·0889
72	0	8·4758	7·8315	79	0	8·6989	8·2503	86	0	9·1430	9·1243
	10	·4801	·8394		10	·7057	·2634		10	·1615	·1612
	20	·4845	·8472		20	·7126	·2767		20	·1809	·1998
	30	·4889	·8552		30	·7196	·2902		30	·2011	·2401
	40	·4933	·8633		40	·7267	·3040		40	·2224	·2825
	50	·4978	·8714		50	·7340	·3180		50	·2447	·3270
73	0	8·5023	7·8796	80	0	8·7413	8·3322	87	0	9·2682	9·3739
	10	·5068	·8879		10	·7487	·3466		10	·2931	·4235
	20	·5114	·8963		20	·7563	·3613		20	·3194	·4761
	30	·5160	·9048		30	·7640	·3763		30	·3475	·5321
	40	·5207	·9134		40	·7718	·3915		40	·3775	·5920
	50	·5254	·9221		50	·7798	·4070		50	·4097	·6564
74	0	8·5301	7·9308	81	0	8·7879	8·4228	88	0	9·4445	9·7259
	10	·5349	·9397		10	·7961	·4389		10	·4823	·8014
	20	·5397	·9486		20	·8045	·4554		20	·5238	·8842
	30	·5446	·9577		30	·8131	·4721		30	·5695	9·9757
	40	·5495	·9669		40	·8218	·4892		40	·6207	0·0779
	50	·5545	·9761		50	·8307	·5066		50	·6787	·1939
75	0	8·5595	7·9855	82	0	8·8398	8·5244	89	0	9·7457	0·3278
	10	·5646	·9950		10	·8491	·5426		10	·8249	·4861
	20	·5698	8·0046		20	·8585	·5611		20	9·9218	·6799
	30	·5749	·0143		30	·8682	·5801		30	0·0467	0·9298
	40	·5802	·0241		40	·8780	·5995		40	·2228	1·2820
	50	·5855	·0341		50	·8881	·6194		50	0·5239	1·8840
76	0	8·5908	8·0441	83	0	8·8985	8·6397	90	0	....	....

TABLE X.

Ann. Prec. in R.A.	C_a.	Ann. Prec. in R.A.	C_a.	Ann. Prec. in Dec.	C_δ.
s.	s.	s.	s.	"	"
−2·0	+0·402	+3·0	+0·190	−20	+0·85
−1·9	·398	3·1	·185	−19	·81
−1·8	·394	3·2	·181	−18	·77
−1·7	+0·389	3·3	+0·177	−17	+0·73
−1·6	·385	3·4	·173	−16	·68
−1·5	·381	3·5	·168	−15	·64
−1·4	+0·377	3·6	+0·164	−14	+0·60
−1·3	·372	3·7	·160	−13	·56
−1·2	·368	3·8	·156	−12	·51
−1·1	+0·364	3·9	+0·151	−11	+0·47
−1·0	·360	4·0	·147	−10	·43
−0·9	·355	4·1	·143	− 9	·39
−0·8	+0·351	4·2	+0·139	− 8	+0·34
−0·7	·347	4·3	·134	− 7	·30
−0·6	·343	4·4	·130	− 6	·26
−0·5	+0·338	4·5	+0·126	− 5	+0·22
−0·4	·334	4·6	·122	− 4	·17
−0·3	·330	4·7	·117	− 3	·13
−0·2	+0·326	4·8	+0·113	− 2	+0·09
−0·1	·321	4·9	·109	− 1	·04
0·0	·317	5·0	·105	0	·00
+0·1	+0·313	5·1	+0·100	+ 1	−0·04
0·2	·309	5·2	·096	2	·09
0·3	·304	5·3	·092	3	·13
0·4	+0·300	5·4	+0·088	4	−0·17
0·5	·296	5·5	·083	5	·22
0·6	·292	5·6	·079	6	·26
0·7	+0·287	5·7	+0·075	7	−0·30
0·8	·283	5·8	·071	8	·34
0·9	·279	5·9	·066	9	·39
1·0	+0·275	6·0	+0·062	10	−0·43
1·1	·270	6·1	·058	11	·47
1·2	·266	6·2	·054	12	·51
1·3	+0·262	6·3	+0·049	13	−0·56
1·4	·258	6·4	·045	14	·60
1·5	·253	6·5	·041	15	·64
1·6	+0·249	6·6	+0·037	16	−0·68
1·7	·245	6·7	·032	17	·73
1·8	·241	6·8	·028	18	·77
1·9	+0·236	6·9	+0·024	19	−0·81
2·0	·232	7·0	·020	+20	−0·85
2·1	·228	7·1	·015		
2·2	+0·224	7·2	+0·011		
2·3	·219	7·3	·007		
2·4	·215	7·4	+0·003		
2·5	+0·211	7·5	−0·002		
2·6	·207	7·6	− ·006		
2·7	·202	7·7	− ·010		
2·8	+0·198	7·8	−0·014		
2·9	·194	7·9	− ·019		
+3·0	+0·190	+8·0	−0·023		

APPENDIX IV.

CONSTANTS AND TABLES FOR THE TRIGONOMETRIC REDUCTION OF MEAN PLACES OF THE STARS.

I.—General Expressions for the Constants of Reduction.

The constants ζ_0, z, and θ, of which the general expressions follow, fix the position of the mean equator and equinox at an epoch T_0+T relative to their positions at an initial epoch T_0. T is the interval between the two epochs, expressed in terms of 100 solar years, or 36524·22 days, as the unit of time.

The geometric meaning of these constants may be gathered from the chapter on Precession, §§ 127-131, in which they are developed. The expressions for initial epochs intermediate between those given can be found by interpolation.

In using the numbers ζ_0, z, and θ to reduce mean places of stars, the latter are supposed to be given for the initial epoch, and the problem is to reduce them to the epoch T_0+T, which may be earlier or later. For cases when the initial epoch is earlier than 1850, the general expressions may be extended by carrying the coefficients of T and its powers backward by means of the uniform centennial change derived from the given coefficients.

We may also, in any case, form the constants by the principle that the two epochs are interchangeable, provided that we also change

$$\zeta_0,\ z,\ \text{and}\ \theta$$

into

$$-z,\ -\zeta_0,\ \text{and}\ -\theta.$$

For example, if we wish to reduce the positions of the Bradley stars from 1755 to 1875, we may form the numbers for the reverse reduction from 1875 as the initial epoch to 1755.

Putting $T = -1{\cdot}20,$

we thus find $\zeta_0 = -2764{\cdot}28,$

$z = -2763{\cdot}14,$

$\theta = -2406{\cdot}42.$

Hence, for reducing from 1755 to 1875, we use

$\zeta_0 = 2763{\cdot}14,$

$z = 2764{\cdot}28,$

$\theta = 2406{\cdot}42.$

To obtain the numbers without this reversal, we find that in ζ_0 the coefficient of T is represented by the expression

$$2303''{\cdot}55 + 1''{\cdot}40(T_0 - 1850).$$

We therefore have, when $T_0 = 1755,$

$$\zeta_0 = 2302''{\cdot}22T + 0''{\cdot}30T^2 + 0''{\cdot}017T^3,$$

which gives, for $T = 1{\cdot}20,$

$$\zeta_0 = 2763''{\cdot}13$$
$$0''{\cdot}79T^2 = \quad 1\ {\cdot}14$$
$$z = 2764\ {\cdot}27$$

It will be seen that these numbers agree with those found by reversal within $0''{\cdot}01$.

This relation between the numbers for the two epochs affords a check upon the correctness of the expressions as printed, because, for example, the numbers found for $T_0 = 1850$, $T = +1$ should correspond to those found for $T_0 = 1950$; $T = -1$. In fact, we find,

1850 to 1950.	1950 to 1850.
$\zeta_0 = 2303{\cdot}87.$	$\zeta_0 = -2304{\cdot}67.$
$z = 2304{\cdot}66.$	$z = -2303{\cdot}88.$
$\theta = 2004{\cdot}64.$	$\theta = -2004{\cdot}65.$

The coefficients of T and its powers in all the expressions increase or diminish uniformly with the time for several centuries. This check can, therefore, always be applied by continuing the expressions to former values of the initial epoch by addition or subtraction.

It will be noted that these three constants are always positive when the initial epoch is the earlier of the two, and negative when it is the later.

General Expressions for the Constants of Reduction.

Initial epoch. T_0	Expression.		
1850	$\zeta_0 = 2303''\cdot55T$	$+ 0''\cdot30T^2$	$+ 0''\cdot017T^3$
1875	2303 ·90	+0 ·30	+0 ·017
1900	2304 ·25	+0 ·30	+0 ·017
1925	2304 ·60	+0 ·30	+0 ·017
1950	2304 ·95	+0 ·30	+0 ·017
1850	$\theta = 2005''\cdot11T$	$- 0''\cdot43T^2$	$- 0''\cdot041T^3$
1875	2004 ·90	−0 ·43	−0 ·041
1900	2004 ·68	−0 ·43	−0 ·041
1925	2004 ·47	−0 ·43	−0 ·041
1950	2004 ·26	−0 ·43	−0 ·041
1850	$\psi = 5136''\cdot82T$	$- 1''\cdot07T^2$	$- 0''\cdot001T^3$
1875	5136 ·95	−1 ·07	−0 ·001
1900	5137 ·07	−1 ·07	−0 ·001
1925	5137 ·20	−1 ·07	−0 ·001
1950	5137 ·32	−1 ·07	−0 ·001
1850	$\lambda = 13''\cdot42T$	$- 2''\cdot38T^2$	$- 0''\cdot003T^3$
1875	12 ·95	−2 ·38	−0 ·003
1900	12 ·48	−2 ·38	−0 ·003
1925	12 ·00	−2 ·38	−0 ·003
1950	11 ·53	−2 ·38	−0 ·003

$$z = \zeta_0 + 0''\cdot79T^2.$$

II.—Special Values for the Usual Epochs.

During the present generation the common equinox to which all star positions are reduced for the purpose of comparison will generally be either 1875 or 1900. We therefore give tables of the special values of ζ_0, z, θ, and m, as derived from the preceding expressions, for reduction from the dates of the principal catalogues of stars to the equinoxes of these two epochs.

Special Values of Constants for the Reduction of Mean Places of the Stars from various dates to the Equinox and Equator of 1875 *and* 1900.

TABLE XIA, FOR REDUCTION TO 1875.

Date.	ζ_0.		z.		θ.		log h sin θ.	m.	
	′	″	′	″	′	″		m.	s.
1755	46	3·14	46	4·28	40	6·92	2·205 27	+6	8·495
1800	28	47·32	28	47·76	25	3·84	2·001 12	3	50·339
1825	19	11·68	19	11·88	16	42·55	1·825 01	2	33·571
1830	17	16·54	17	16·70	15	2·29	1·779 25	2	18·216
1840	13	26·23	13	26·33	11	41·76	1·670 10	1	47·504
1845	11	31·07	11	31·14	10	1·51	1·603 15	1	32·147
1850	9	35·91	9	35·96	8	21·25	1·523 96	1	16·791
1855	7	40·74	7	40·77	6	41·00	1·427 04	1	1·434
1860	5	45·56	5	45·58	5	0·74	1·302 11	0	46·076
1865	3	50·38	3	50·39	3	20·49	1·126 01	0	30·718
1870	1	55·19	1	55·19	1	40·25	0·824 98	0	15·359
1875	0	0·00	0	0·00	0	0·00		0	0·000
1880	– 1	55·20	– 1	55·20	– 1	40·24	0·824 96$_n$	–0	15·360
1885	– 3.	50·40	– 3	50·39	– 3	20·48	1·125 99$_n$	–0	30·720
1890	– 5	45·61	– 5	45·59	– 5	0·72	1·302 08$_n$	–0	46·080
1895	– 7	40·82	– 7	40·79	– 6	40·96	1·427 01$_n$	– 1	1·441
1900	– 9	36·04	– 9	35·99	– 8	21·20	1·523 92$_n$	– 1	16·802
1905	–11	31·27	–11	31·20	–10	1·43	1·603 09$_n$	– 1	32·164
1910	–13	26·50	–13	26·40	–11	41·66	1·670 04$_n$	– 1	47·527
1915	–15	21·74	–15	21·61	–13	21·89	1·728 02$_n$	–2	2·890
1920	–17	16·98	–17	16·82	–15	2·11	1·779 17$_n$	–2	18·253
1925	–19	12·22	–19	12·03	–16	42·34	1·824 92$_n$	–2	33·617
1930	–21	7·48	–21	7·24	–18	22·56	1·866 31$_n$	–2	48·981
1935	–23	2·74	–23	2·45	–20	2·78	1·904 09$_n$	–3	4·346
1940	–24	58·00	–24	57·67	–21	42·99	1·958 85$_n$	–3	19·711
1945	–26	53·27	–26	52·88	–23	23·20	1·971 03$_n$	–3	35·077
1950	–28	48·55	–28	48·10	–25	3·42	2·000 98$_n$	–3	50·443

TABLE XIb., FOR REDUCTION TO 1900.

Date.	ζ_0.		z.		θ.		log h sin θ.	m.	
	′	″	′	″	′	″		m.	s.
1755	55	38·92	55	40·58	48	27·56	2·287 42	+7	25·300
1800	38	23·18	38	23·96	33	25·06	2·126 03	5	7·143
1825	28	47·58	28	48·02	25	3·73	2·001 08	3	50·373
1830	26	52·45	26	52·83	23	23·47	1·971 11	3	35·019
1835	24	57·31	24	57·64	21	43·21	1·938 92	3	19·663
1840	23	2·16	23	2·44	20	2·95	1·904 16	3	4·307
1845	21	7·01	21	7·25	18	22·70	1·866 36	2	48·951
1850	19	11·85	19	12·05	16	42·44	1·824 97	2	33·593
1855	17	16·69	17	16·85	15	2·19	1·779 21	2	18·236
1860	15	21·53	15	21·65	13	21·94	1·728 05	2	2·879
1865	13	26·35	13	26·45	11	41·69	1·670 05	1	47·520
1870	11	31·18	11	31·25	10	1·44	1·603 10	1	32·162
1875	9	35·99	9	36·04	8	21·20	1·523 92	1	16·802
1880	7	40·81	7	40·84	6	40·95	1·427 00	1	1·443
1885	5	45·61	5	45·63	5	0·71	1·302 06	0	46·083
1890	3	50·41	3	50·42	3	20·47	1·125 96	0	30·722
1895	1	55·21	1	55·21	1	40·24	0·824 93	0	15·361
1900	0	0·00	0	0·00	0	0·00		0	0·000
1905	− 1	55·21	− 1	55·22	− 1	40·23	$0{\cdot}824\ 92_n$	−0	15·362
1910	− 3	50·44	− 3	50·43	− 3	20·46	$1{\cdot}125\ 94_n$	−0	30·725
1915	− 5	45·66	− 5	45·64	− 5	0·69	$1{\cdot}302\ 03_n$	−0	46·087
1920	− 7	40·89	− 7	40·86	− 6	40·92	$1{\cdot}426\ 96_n$	−1	1·450
1925	− 9	36·13	− 9	36·08	− 8	21·14	$1{\cdot}523\ 87_n$	−1	16·814
1930	−11	31·37	−11	31·30	−10	1·36	$1{\cdot}603\ 04_n$	−1	32·178
1935	−13	26·62	−13	26·52	−11	41·58	$1{\cdot}669\ 99_n$	−1	47·543
1940	−15	21·87	−15	21·75	−13	21·80	$1{\cdot}727\ 98_n$	−2	2·908
1945	−17	17·13	−17	16·97	−15	2·02	$1{\cdot}779\ 12_n$	−2	18·273
1950	−19	12·40	−19	12·20	−16	42·23	$1{\cdot}824\ 87_n$	−2	33·640

The only constants usually necessary are ζ_0, $h \sin \theta$, and $m = \zeta_0 + z$. There are two cases between which the choice depends on the nearness of the star to the pole, the length of time over which the reduction extends, and the degree of numerical precision required. If the quotient by dividing the interval of reduction in years by the north polar distance in degrees does not exceed 30, we may, practically, nearly always use method A below, which requires the approximate value of p called p_0 in § 140, a correction being applied by means of the Tables X., XVI., and XVII. When using this method, Tables XII. and XIII. are not necessary. Within 30° of the pole we may consider the thousandths of a second as unimportant, unless an unusual degree of theoretical precision is required.

A.—The Usual Method.

Putting α_0 and δ_0 for the given R.A. and Dec. for epoch T_0 we compute

$$a = \alpha_0 + \zeta_0.$$

If α_0 is reduced to arc, it will suffice to express α_0 and ζ_0 to hundredths of a minute, reducing the seconds of ζ_0 to minutes. If we have a table of logarithms, (5-place) with argument in time, we may use, without important error,

$$a = \alpha_0 + \tfrac{1}{2}\text{m}.$$

Form the logarithm of

$$p_s = h \sin\theta \tan\delta_0.$$

Enter Table XIV. with Arg. $\log p_s \cos a$ and take out $\log K$, which has the same algebraic sign as $p_s \cos a$, and is to be taken from the column + or − according to this sign.

Compute $$\Delta_0 a = K p_s \sin a.$$

Take $\Delta_1 a$ from Table XV. with the elapsed time in years, $\equiv T_y$, and a as the arguments. If T_y is not found in the table, note that for any a, $\Delta_1 a$ is proportional to its square and may be found by multiplying the value of $\Delta_1 a$ for $T_y = 100Y$ by T^2.

Take the factor F from Table XVI., and form

$$\Delta_2 a = F\Delta_1 a.$$

These numbers are so small that this product may be formed mentally at sight. The third decimal of F is practically unnecessary when $F > 0{\cdot}100$, and may nearly always be dropped.

Take the reduction R from tangent to arc from Table XIII. with argument $\Delta_0 a + \Delta_1 a + \Delta_2 a = \Delta_t a$, instead of which we may nearly always use $\Delta_0 a$, without important error. When the argument falls in the first column, the value of R is the same for all intermediate values of the argument before the next one following.

Then $$\Delta a = \Delta_0 a + \Delta_1 a + \Delta_2 a - R,$$

noting that R is always numerically subtractive.

Should the value of Δa exceed that to which the table extends, we subtract $\log h$ from $\log \Delta_t a$, which will give the tangent of Δa.

$$\tan \Delta a = [5{\cdot}861\ 666]\ \Delta_t a.$$

With this tangent Δa may be taken from an ordinary table of log. tangents. Then

$$\alpha = \alpha_0 + \Delta a + \text{m} = a + z + \Delta a,$$

and if $\Delta a < 6°$,
$$\delta = \delta_0 + \theta \cos(a + \tfrac{1}{2}\Delta a) \sec \tfrac{1}{2}\Delta a.$$

B.—Rigorous Method.

If $T_y \div \text{N.P.D.}° > 30$, Tables XIV. to XVII. cannot always be used. In this case Method A will suffice for a reduction which shall be accurate to $\pm 0{\cdot}001$ s. only when $T_y \sin \alpha \tan \delta < 40$, and will fail when $T_y \sin \alpha \tan \delta > 40$. When, from this cause, Method A is not applicable, we modify it as follows:

Compute
$$a = \alpha_0 + \zeta_0,$$
$$p = \sin \theta (\tan \delta_0 + \tan \tfrac{1}{2}\theta \cos a).$$

Log tan $\frac{1}{2}\theta$ may be taken from Table XII. or computed. As the usual tables of addition and subtraction logarithms are not convenient for this computation, we give Table XIII. We form

$$\text{Diff.} = \log \tan \delta_0 - \log \tan \tfrac{1}{2}\theta \cos a.$$

With this argument, take from Table XIII. a logarithm from the column Add. to add to $\log \tan \delta_0$ when $\tan \delta_0$ and $\tan \frac{1}{2}\theta \cos \alpha$ are of the same sign, or from the column Subt. to subtract when they are of opposite signs.

We then have
$$\tan \Delta a = \frac{p \sin a}{1 - p \cos a} = Kp \sin a,$$

from which we compute a and α as in Method A.

When $\log p \cos a < 9{\cdot}26$, and 5-place logarithms are sufficiently exact, which will generally be the case, we may use Table XIV. to find $\log(1 - p \cos a)$, entering it with $\log p_s \cos a = [4{\cdot}138\ 33]\, p \cos a$ as the argument. If $p \cos a$ is positive, we have

$$\log \tan \Delta a = \log p \sin a + \log K +$$

if negative,
$$\log \tan \Delta a = \log p \sin a - \log K - .$$

The declination may be reduced as in Method A, or by the rigorous formula (19), § 138.

TABLE XII.

log tan $\frac{1}{2}\theta$. *Arg.:—Number of Years through which the Reduction extends.*

The tangent is positive for a reduction from an earlier to a later epoch, and negative in the opposite case.

Years.	log tan $\frac{1}{2}\theta$.	Years.	log tan $\frac{1}{2}\theta$.	Years.	log tan $\frac{1}{2}\theta$.	Years.	log tan $\frac{1}{2}\theta$.	Years.	log tan $\frac{1}{2}\theta$.
20	6·9877	45	7·3398	70	7·5317	95	7·6643	120	7·7658
25	7·0845	50	7·3856	75	7·5617	100	7·6866	125	7·7835
30	7·1636	55	7·4270	80	7·5898	105	7·7078	130	7·8006
35	7·2306	60	7·4647	85	7·6161	110	7·7280	135	7·8170
40	7·2887	65	7·4995	90	7·6408	115	7·7473	140	7·8330

Addition and Subtraction Logarithms to the Sixth Decimal Place.

TABLE XIII.

Diff.	Add.	Subt.	Diff.	Add.	Subt.	Diff.	A. or S.	Diff.	A. or S.
							·000		·000
2·20	$0·002732_{62}$	2749_{63}	2·60	$0·001090_{25}$	1092_{25}	3·00	434_{10}	3·40	173_{4}
·21	2670_{61}	2686_{61}	·61	1065_{24}	1067_{24}	·01	424_{9}	·41	169_{4}
·22	2609_{59}	2625_{60}	·62	1041_{24}	1043_{24}	·02	415_{10}	·42	165_{4}
·23	2550_{58}	2565_{59}	·63	1017_{23}	1019_{23}	·03	405_{9}	·43	161_{3}
·24	2492_{57}	2506_{57}	·64	0994_{23}	0996_{23}	·04	396_{9}	·44	158_{4}
2·25	$0·002435_{55}$	2449_{56}	2·65	$0·000971_{22}$	0973_{22}	3·05	387_{9}	3·45	154_{3}
·26	2380_{54}	2393_{54}	·66	0949_{21}	0951_{22}	·06	378_{8}	·46	151_{4}
·27	2326_{53}	2239_{54}	·67	0928_{22}	0929_{21}	·07	370_{9}	·47	147_{3}
·28	2273_{51}	2285_{52}	·68	0906_{20}	0908_{20}	·08	361_{8}	·48	144_{3}
·29	2222_{51}	2233_{51}	·69	0886_{20}	0888_{21}	·09	353_{8}	·49	141_{4}
2·30	$0·002171_{49}$	2182_{50}	2·70	$0·000866_{20}$	0867_{19}	3·10	345_{8}	3·50	137_{3}
·31	2122_{48}	2132_{48}	·71	0846_{19}	0848_{20}	·11	337_{8}	·51	134_{3}
·32	2074_{47}	2084_{48}	·72	0827_{19}	0828_{19}	·12	329_{7}	·52	131_{3}
·33	2027_{46}	2036_{46}	·73	0808_{18}	0809_{18}	·13	322_{7}	·53	128_{3}
·34	1981_{45}	1990_{46}	·74	0790_{18}	0791_{18}	·14	315_{8}	·54	125_{3}
2·35	$0·001936_{44}$	1944_{44}	2·75	$0·000772_{18}$	0773_{18}	3·15	307_{7}	3·55	122_{13}
·36	1892_{43}	1900_{43}	·76	0754_{17}	0755_{17}	·16	300_{6}	·60	109_{22}
·37	1849_{42}	1857_{43}	·77	0737_{17}	0738_{17}	·17	294_{7}	·70	087_{18}
·38	1807_{41}	1814_{41}	·78	0720_{16}	0721_{16}	·18	287_{7}	·80	069_{14}
·39	1766_{40}	1773_{41}	·79	0704_{16}	0705_{16}	·19	280_{6}	·90	055_{12}
2·40	$0·001726_{40}$	1732_{39}	2·80	$0·000688_{16}$	0689_{16}	3·20	274_{6}	4·00	043_{9}
·41	1686_{38}	1693_{39}	·81	0672_{15}	0673_{15}	·21	268_{6}	·10	034_{7}
·42	1648_{37}	1654_{37}	·82	0657_{15}	0658_{15}	·22	262_{6}	·20	027_{5}
·43	1611_{37}	1617_{37}	·83	0642_{15}	0643_{15}	·23	256_{6}	·30	022_{5}
·44	1574_{36}	1580_{36}	·84	0627_{14}	0628_{14}	·24	250_{6}	·40	017_{3}
2·45	$0·001538_{35}$	1544_{36}	2·85	$0·000613_{14}$	0614_{14}	3·25	244_{5}	4·50	014_{3}
·46	1503_{34}	1508_{34}	·86	0599_{14}	0600_{14}	·26	239_{6}	·60	011_{2}
·47	1469_{33}	1474_{34}	·87	0585_{13}	0586_{13}	·27	233_{5}	·70	009_{2}
·48	1436_{33}	1440_{32}	·88	0572_{13}	0573_{13}	·28	228_{5}	·80	007_{2}
·49	1403_{32}	1408_{32}	·89	0559_{13}	0560_{13}	·29	223_{5}	·90	005_{1}
2·50	$0·001371_{31}$	1376_{32}	2·90	$0·000546_{12}$	0547_{12}	3·30	218_{5}	5·00	004_{1}
·51	1340_{30}	1344_{30}	·91	0534_{12}	0535_{13}	·31	213_{5}	·10	003_{0}
·52	1310_{30}	1314_{30}	·92	0522_{12}	0522_{11}	·32	208_{5}	·20	003_{1}
·53	1280_{29}	1284_{30}	·93	0510_{12}	0511_{12}	·33	203_{5}	·30	002_{0}
·54	1251_{29}	1254_{28}	·94	0498_{11}	0499_{11}	·34	198_{4}	·40	002_{1}
2·55	$0·001222_{27}$	1226_{28}	2·95	$0·000487_{11}$	0488_{12}	3·35	194_{4}	5·50	001_{0}
·56	1195_{28}	1198_{28}	·96	0476_{11}	0476_{10}	·36	190_{5}	·60	001_{0}
·57	1167_{26}	1170_{26}	·97	0465_{10}	0466_{11}	·37	185_{4}	·70	001_{0}
·58	1141_{26}	1144_{26}	·98	0455_{11}	0455_{10}	·38	181_{4}	·80	001_{0}
·59	1115_{25}	1118_{26}	·99	0444_{10}	0445_{10}	·39	177_{4}	·90	001_{1}
2·60	0·001090	1092	3·00	0·000434	0435	3·40	173	6·00	000

TABLE XIV.

$$\text{Argument} = \log p, \cos a\,; \quad K = \frac{1}{1 - p\cos a}.$$

Arg.	log *K*. Arg. +	log *K*. Arg. −	Arg.	log *K*. Arg. +	log *K*. Arg. −	Arg.	log *K*. Arg. +	log *K*. Arg. −
	+	−		+	−		+	−
9·20	0·00 001	0·00 001	1·40	0·00 079	0·00 079	1·80	0·00 200	0·00 199
·30	001	001	·41	081	081	·81	204	204
·40	001	001	·42	083	083	·82	209	208
·50	001	001	·43	085	085	·83	214	213
·60	001	001	·44	087	087	·84	219	218
9·70	0·00 002	0·00 002	1·45	0·00 089	0·00 089	1·85	0·00 224	0·00 223
·80	002	002	·46	091	091	·86	229	228
·90	003	003	·47	093	093	·87	235	234
0·00	003	003	·48	096	095	·88	240	239
·10	004	004	·49	098	098	·89	246	244
0·20	0·00 005	0·00 005	1·50	0·00 100	0·00 100	1·90	0·00 252	0·00 250
·30	006	006	·51	102	102	·91	257	256
·40	008	008	·52	105	104	·92	264	262
·50	010	010	·53	107	107	·93	270	268
·60	013	013	·54	110	109	·94	276	274
0·70	0·00 016	0·00 016	1·55	0·00 112	0·00 112	1·95	0·00 282	0·00 281
·80	020	020	·56	115	115	·96	289	287
·90	025	025	·57	118	117	·97	296	294
1·00	032	032	·58	120	120	·98	303	301
·10	040	040	·59	123	123	·99	310	308
1·20	0·00 050	0·00 050	1·60	0·00 126	0·00 126	2·00	0·00 317	0·00 315
·21	051	051	·61	129	129	·01	324	322
·22	052	052	·62	132	132	·02	332	329
·23	054	054	·63	135	134	·03	340	337
·24	055	055	·64	138	138	·04	348	345
1·25	0·00 056	0·00 056	1·65	0·00 141	0·00 141	2·05	0·00 356	0·00 353
·26	058	058	·66	145	144	·06	364	361
·27	059	059	·67	148	147	·07	373	369
·28	060	060	·68	151	151	·08	381	378
·29	062	062	·69	155	154	·09	390	387
1·30	0·00 063	0·00 063	1·70	0·00 159	0·00 158	2·10	0·00 399	0·00 396
·31	065	065	·71	162	162	·11	409	405
·32	066	066	·72	166	165	·12	418	414
·33	068	068	·73	170	169	·13	428	424
·34	069	069	·74	174	173	·14	438	434
1·35	0·00 071	0·00 071	1·75	0·00 178	0·00 177	2·15	0·00 448	0·00 444
·36	072	072	·76	182	181	·16	459	454
·37	074	074	·77	186	186	·17	470	465
·38	076	076	·78	191	190	·18	481	475
·39	078	078	·79	195	194	·19	492	486
1·40	0·00 079	0·00 079	1·80	0·00 200	0·00 199	2·20	0·00 503	0·00 498

TABLE XIV.—*Continued.*

Argument = $\log p_{,} \cos a$; $K = \dfrac{1}{1 - p \cos a}$.

Arg.	log *K*. Arg. +	log *K*. Arg. −	Arg.	log *K*. Arg. +	log *K*. Arg. −	Arg.	log *K*. Arg. +	log *K*. Arg. −
	+	−		+	−		+	−
2·20	0·00 503	0·00 498	2·60	0·01 276	0·01 240	3·000	0·03 279	0·03 049
·21	515	509	·61	306	268	·001	287	056
·22	527	521	·62	337	297	·002	295	062
·23	540	533	·63	369	327	·003	303	069
·24	552	545	·64	401	357	·004	311	076
2·25	0·00 565	0·00 558	2·65	0·01 434	0·01 388	3·005	0·03 318	0·03 083
·26	579	571	·66	468	420	·006	326	090
·27	592	584	·67	503	453	·007	334	097
·28	606	598	·68	539	486	·008	342	103
·29	620	612	·69	575	520	·009	350	110
2·30	0·00 635	0·00 626	2·70	0·01 612	0·01 555	3·010	0·03 358	0·03 117
·31	650	640	·71	651	590	·011	366	124
·32	665	655	·72	690	627	·012	375	131
·33	681	670	·73	730	664	·013	383	138
·34	696	686	·74	771	702	·014	391	145
2·35	0·00 713	0·00 701	2·75	0·01 813	0·01 741	3·015	0·03 399	0·03 152
·36	730	718	·76	857	780	·016	407	159
·37	747	734	·77	901	821	·017	415	166
·38	764	751	·78	946	862	·018	423	173
·39	782	768	·79	992	905	·019	432	180
2·40	0·00 801	0·00 786	2·80	0·02 040	0·01 948	3·020	0·03 440	0·03 187
·41	819	804	·81	089	993	·021	448	194
·42	839	823	·82	138	0·02 038	·022	456	201
·43	858	842	·83	190	084	·023	465	209
·44	879	861	·84	242	132	·024	473	216
2·45	0·00 899	0·00 881	2·85	0·02 296	0·02 180	3·025	0·03 481	0·03 223
·46	921	901	·86	350	230	·026	490	230
·47	942	922	·87	407	280	·027	498	237
·48	964	943	·88	464	332	·028	506	244
·49	987	965	·89	524	385	·029	515	252
2·50	0·01 010	0·00 987	2·90	0·02 584	0·02 439	3·030	0·03 523	0·03 259
·51	034	0·01 010	·91	646	494	·031	532	266
·52	059	033	·92	710	551	·032	540	273
·53	084	057	·93	775	608	·033	549	281
·54	109	082	·94	842	667	·034	557	288
2·55	0·01 135	0·01 106	2·95	0·02 910	0·02 727	3·035	0·03 566	0·03 295
·56	162	132	·96	980	789	·036	574	302
·57	190	158	·97	0·03 052	852	·037	583	310
·58	218	184	·98	126	916	·038	592	317
·59	246	212	·99	202	982	·039	600	324
2·60	0·01 276	0·01 240	3·00	0·03 279	0·03 049	3·040	0·03 609	0·03 332

TABLE XIV.—*Continued.*

$$\text{Argument} = \log p_s \cos a\,; \quad K = \frac{1}{1 - p \cos a}.$$

Arg.	log *K*. Arg. +	log *K*. Arg. −	Arg.	log *K*. Arg. +	log *K*. Arg. −	Arg.	log *K*. Arg. +	log *K*. Arg. −
	+	−		+	−		+	−
3·040	0·03 609	0·03 332	3·080	0·03 973	0·03 640	3·120	0·04 377	0·03 976
·041	618	339	·081	983	648	·121	387	985
·042	626	347	·082	993	656	·122	398	993
·043	635	354	·083	0·04 002	664	·123	409	0·04 002
·044	644	362	·084	012	672	·124	419	011
3·045	0·03 653	0·03 369	3·085	0·04 022	0·03 681	3·125	0·04 430	0·04 020
·046	661	376	·086	031	689	·126	441	029
·047	670	384	·087	041	697	·127	452	037
·048	679	391	·088	051	705	·128	462	046
·049	688	399	·089	061	713	·129	473	055
3·050	0·03 697	0·03 406	3·090	0·04 070	0·03 721	3·130	0·04 484	0·04 064
·051	706	414	·091	080	730	·131	495	073
·052	714	422	·092	090	738	·132	506	082
·053	723	429	·093	100	746	·133	517	091
·054	732	437	·094	110	754	·134	528	100
3·055	0·03 741	0·03 444	3·095	0·04 120	0·03 763	3·135	0·04 539	0·04 109
·056	750	452	·096	130	771	·136	550	118
·057	759	460	·097	140	779	·137	561	127
·058	768	467	·098	150	788	·138	572	136
·059	777	475	·099	160	796	·139	583	145
3·060	0·03 787	0·03 483	3·100	0·04 170	0·03 804	3·140	0·04 594	0·04 154
·061	796	490	·101	180	813	·141	605	164
·062	805	498	·102	190	821	·142	617	173
·063	814	506	·103	200	830	·143	628	182
·064	823	514	·104	210	838	·144	639	191
3·065	0·03 832	0·03 521	3·105	0·04 221	0·03 847	3·145	0·04 650	0·04 200
·066	842	529	·106	231	855	·146	662	210
·067	851	537	·107	241	864	·147	673	219
·068	860	545	·108	251	872	·148	684	228
·069	869	553	·109	262	881	·149	696	237
3·070	0·03 879	0·03 561	3·110	0·04 272	0·03 889	3·150	0·04 707	0·04 247
·071	888	569	·111	282	898	·151	719	256
·072	898	576	·112	293	906	·152	730	265
·073	907	584	·113	303	915	·153	742	275
·074	916	592	·114	314	924	·154	753	284
3·075	0·03 926	0·03 600	3·115	0·04 324	0·03 932	3·155	0·04 765	0·04 293
·076	935	608	·116	335	941	·156	777	303
·077	945	616	·117	345	950	·157	788	312
·078	954	624	·118	356	958	·158	800	322
·079	964	632	·119	366	967	·159	812	331
3·080	0·03 973	0·03 640	3·120	0·04 377	0·03 976	3·160	0·04 823	0·04 341

TABLE XIV.—*Continued.*

Argument $= \log p_s \cos a$; $K = \dfrac{1}{1 - p\cos a}$.

Arg.	log K. Arg. +	log K. Arg. −	Arg.	log K. Arg. +	log K. Arg. −	Arg.	log K. Arg. +	log K. Arg. −
	+	−		+	−		+	−
3·160	0·04 823	0·04 341	3·200	0·05 318	0·04 737	3·240	0·05 868	0·05 168
·161	835	350	·201	331	748	·241	882	180
·162	847	360	·202	344	758	·242	897	191
·163	859	369	·203	358	769	·243	911	202
·164	870	379	·204	371	779	·244	926	213
3·165	0·04 882	0·04 389	3·205	0·05 384	0·04 789	3·245	0·05 940	0·05 225
·166	894	398	·206	397	800	·246	955	236
·167	906	408	·207	410	810	·247	970	247
·168	918	418	·208	424	821	·248	984	259
·169	930	427	·209	437	831	·249	999	270
3·170	0·04 942	0·04 437	3·210	0·05 450	0·04 842	3·250	0·06 014	0·05 282
·171	954	447	·211	464	852	·251	029	293
·172	966	456	·212	477	862	·252	044	305
·173	979	466	·213	491	874	·253	059	316
·174	991	476	·214	504	884	·254	074	328
3·175	0·05 003	0·04 486	3·215	0·05 518	0·04 895	3·255	0·06 089	0·05 339
·176	015	496	·216	531	906	·256	104	351
·177	027	505	·217	545	916	·257	119	362
·178	040	515	·218	558	927	·258	134	374
·179	052	525	·219	572	938	·259	149	386
3·180	0·05 064	0·04 535	3·220	0·05 586	0·04 949	3·260	0·06 165	0·05 397
·181	077	545	·221	599	959	·261	180	409
·182	089	555	·222	613	970	·262	195	421
·183	102	565	·223	627	981	·263	211	433
·184	114	575	·224	641	992	·264	226	444
3·185	0·05 127	0·04 585	3·225	0·05 655	0·05 003	3·265	0·06 241	0·05 456
·186	139	595	·226	669	014	·266	257	468
·187	152	605	·227	683	024	·267	272	480
·188	164	615	·228	697	035	·268	288	492
·189	177	625	·229	711	046	·269	304	504
3·190	0·05 190	0·04 635	3·230	0·05 725	0·05 057	3·270	0·06 319	0·05 515
·191	202	645	·231	739	068	·271	335	527
·192	215	655	·232	753	079	·272	351	539
·193	228	666	·233	767	090	·273	366	551
·194	241	676	·234	781	101	·274	382	563
3·195	0·05 254	0·04 686	3·235	0·05 796	0·05 113	3·275	0·06 398	0·05 575
·196	266	696	·236	810	124	·276	414	587
·197	279	707	·237	825	135	·277	430	599
·198	292	717	·238	839	146	·278	446	612
·199	305	727	·239	853	157	·279	462	624
3·200	0·05 318	0·04 737	3·240	0·05 868	0·05 168	3·280	0·06 478	0·05 636

TABLE XIV.—*Concluded.*

$$\text{Argument} = \log p_s \cos a\,; \quad K = \frac{1}{1 - p\cos a}.$$

Arg.	log *K*. Arg. +	log *K*. Arg. −	Arg.	log *K*. Arg. +	log *K*. Arg. −	Arg.	log *K*. Arg. +	log *K*. Arg. −
	+	−		+	−		+	−
3·280	0·06 478	0·05 636	3·320	0·07 157	0·06 143	3·360	0·07 915	0·06 692
·281	494	648	·321	175	156	·361	935	706
·282	510	660	·322	193	169	·362	955	721
·283	526	673	·323	211	183	·363	975	735
·284	543	685	·324	229	196	·364	995	749
3·285	0·06 559	0·05 697	3·325	0·07 247	0·06 209	3·365	0·08 015	0·06 764
·286	575	709	·326	266	223	·366	035	778
·287	592	722	·327	284	236	·367	056	793
·288	608	734	·328	302	249	·368	076	807
·289	624	746	·329	320	263	·369	097	822
3·290	0·06 641	0·05 759	3·330	0·07 339	0·06 276	3·370	0·08 117	0·06 836
·291	658	771	·331	357	290	·371	138	851
·292	674	784	·332	376	303	·372	158	865
·293	691	796	·333	394	317	·373	179	880
·294	707	809	·334	413	330	·374	200	895
3·295	0·06 724	0·05 821	3·335	0·07 432	0·06 344	3·375	0·08 221	0·06 909
·296	741	834	·336	450	357	·376	242	924
·297	758	846	·337	469	371	·377	262	939
·298	775	859	·338	488	385	·378	283	954
·299	792	872	·339	507	398	·379	304	969
3·300	0·06 808	0·05 884	3·340	0·07 525	0·06 412	3·380	0·08 326	0·06 983
·301	825	897	·341	544	426	·381	347	998
·302	843	910	·342	563	440	·382	368	0·07 013
·303	860	922	·343	582	453	·383	389	028
·304	877	935	·344	602	467	·384	411	043
3·305	0·06 894	0·05 948	3·345	0·07 621	0·06 481	3·385	0·08 432	0·07 058
·306	911	961	·346	640	495	·386	453	073
·307	928	974	·347	659	509	·387	475	088
·308	946	987	·348	679	523	·388	496	103
·309	963	999	·349	698	537	·389	518	118
3·310	0·06 980	0·06 012	3·350	0·07 717	0·06 551	3·390	0·08 540	0·07 133
·311	998	025	·351	737	565	·391	562	149
·312	0·07 015	038	·352	756	579	·392	583	164
·313	033	051	·353	776	593	·393	605	179
·314	051	064	·354	796	607	·394	627	194
3·315	0·07 068	0·06 077	3·355	0·07 815	0·06 621	3·395	0·08 649	0·07 209
·316	086	090	·356	835	635	·396	671	225
·317	104	103	·357	855	649	·397	693	240
·318	122	117	·358	875	664	·398	716	255
·319	139	130	·359	895	678	·399	738	271
3·320	0·07 157	0·06 143	3·360	0·07 915	0·06 692	3·400	0·08 760	0·07 286

TABLE XV.

$\Delta_1 a$. Arg. a, or $a - 12\,h$. Arg. at top, $\pm T_y$.

Arg. h.	Arg. m.	25 Y. s.	30 Y. s.	40 Y. s.	50 Y. s.	60 Y. s.	70 Y. s.	80 Y. s.	100 Y. s.	120 Y. s.	145 Y. s.	Arg. m.	Arg. h.
0	0	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0	12
	10	·002	·002	·004	·007	·010	·014	·018	·028	·040	·059	50	
	20	·004	·005	·009	·014	·020	·027	·036	·056	·080	·117	40	
	30	·005	·008	·013	·021	·030	·041	·054	·083	·119	·174	30	
	40	·007	·010	·017	·027	·039	·054	·072	·109	·157	·230	20	
	50	·008	·012	·022	·034	·048	·067	·088	·135	·194	284	10	
1	0	0·010	0·014	0·026	0·040	0·057	0·079	0·104	0·160	0·230	0·336	0	11
	10	·012	·016	·029	·046	·066	·091	·120	·184	·264	·386	50	
	20	·013	·018	·033	·051	·074	·102	·134	·206	·296	·433	40	
	30	·014	·020	·036	·057	·081	·112	·148	·226	·325	·476	30	
	40	·015	·022	·039	·062	·088	·121	·160	·245	·352	·516	20	
	50	·016	·024	·042	·066	·094	·129	·171	·262	·377	·551	10	
2	0	0·017	0·025	0·044	0·069	0·100	0·137	0·181	0·277	0·398	0·583	0	10
	10	·018	·026	·046	·072	·104	·143	·189	·290	·417	·610	50	
	20	·019	·027	·048	·075	·108	·148	·196	·301	·432	·632	40	
	30	·019	·028	·049	·077	·111	·153	·202	·309	·444	·650	30	
	40	·020	·029	·050	·079	·113	·156	·206	·315	·453	·663	20	
	50	·020	·029	·051	·080	·115	·157	·208	·319	·458	·670	10	
3	0	0·020	0·029	0·051	0·080	0·115	0·158	0·209	0·320	0·460	0·673	0	9
	10	·020	·029	·051	·080	·115	·157	·208	·319	·458	·670	50	
	20	·020	·029	·050	·079	·113	·156	·206	·315	·453	·663	40	
	30	·019	·028	·049	·077	·111	·153	·202	·309	·444	·650	30	
	40	·019	·027	·048	·075	·108	·148	·196	·301	·432	·632	20	
	50	·018	·026	·046	·072	·104	·143	·189	·290	·417	·610	10	
4	0	0·017	0·025	0·044	0·069	0·100	0·137	0·181	0·277	0·398	0·583	0	8
	10	·016	·024	·042	·066	·094	·129	·171	·262	·377	·551	50	
	20	·015	·022	·039	·062	·088	·121	·160	·245	·352	·516	40	
	30	·014	·020	·036	·057	·081	·112	·148	·226	·325	·476	30	
	40	·013	·018	·033	·051	·074	·102	·134	·206	·296	·433	20	
	50	·012	·016	·029	·046	·066	·091	·120	·184	·264	·386	10	
5	0	0·010	0·014	0·026	0·040	0·057	0·079	0·104	0·160	0·230	0·336	0	7
	10	·008	·012	·022	·034	·048	·067	·088	·135	·194	·284	50	
	20	·007	·010	·017	·027	·039	·054	·072	·109	·157	·230	40	
	30	·005	·008	·013	·021	·030	·041	·054	·083	·119	·174	30	
	40	·004	·005	·009	·014	·020	·027	·036	·056	·080	·117	20	
	50	·002	·002	·004	·007	·010	·014	·018	·028	·040	·059	10	
6	0	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0·000	0	6

The correction $\Delta_1 a$ is positive when the argument is on the left, and negative when on the right. The sign is the same whether T_y is positive or negative.

TABLE XVI.

To form $\Delta_2 a = \Delta_1 a \times F$. Arg. $\log p_s \cos a$.

Arg.	F. Arg. +	F. Arg. −	Arg.	F. Arg. +	F. Arg. −	Arg.	F. Arg. +	F. Arg. −
1·80	+·009	−·009	3·00	+·163	−·131	3·20	+·278	−·196
1·90	·012	·012	·01	·167	·133	·21	·285	·200
2·00	·015	·015	·02	·172	·136	·22	·293	·204
·10	·019	·018	·03	·176	·139	·23	·302	·208
·20	·023	·023	·04	·181	·142	·24	·310	·212
2·30	+·030	−·029	3·05	+·186	−·145	3·25	+·319	−·216
·40	·038	·036	·06	·191	·148	·26	·328	·220
·50	·048	·045	·07	·196	·151	·27	·337	·224
·60	·060	·055	·08	·201	·154	·28	·347	·228
·70	·077	·069	·09	·206	·158	·29	·357	·232
2·80	+·098	−·086	3·10	+·212	−·161	3·30	+·368	−·237
·83	·106	·091	·11	·217	·164	·31	·379	·241
·86	·115	·097	·12	·223	·168	·32	·390	·246
·89	·124	·104	·13	·229	·171	·33	·402	·250
·92	·133	·110	·14	·235	·175	·34	·414	·255
·95	+·143	−·118	3·15	+·242	−·178	3·35	+·427	−·260
·96	·147	·120	·16	·249	·182	·36	·440	·265
·97	·151	·123	·17	·256	·185	·37	·454	·271
·98	·155	·126	·18	·263	·189	·38	·468	·278
·99	·159	·128	·19	·270	·192	·39	·482	·285
3·00	+·163	−·131	3·20	+·278	−·196	3·40	+·496	−·292

TABLE XVII.

Red. from tan. to arc. Arg. $= \dot{\Delta}_0 a + \Delta_1 a + \Delta_2 a \equiv \Delta_t a$.

$\Delta_t a$.	R.	$\Delta_t a$	R.	$\Delta_t a$.	R.	$\Delta_t a$.	R.	$\Delta_t a$.	R.
s.	s.	s.	s.	s.	s.	s.	s.	s.	s.
76	·001	200	·014	400	·113	600	·381	800	·901
96	·002	210	·016	410	·121	610	·402	810	·936
112	·003	220	·019	420	·129	620	·423	820	·971
126	·004	230	·022	430	·139	630	·443	830	1·007
136	·005	240	·024	440	·150	640	·464	840	1·044
146	·006	250	·028	450	·160	650	·485	850	1·081
155	·007	260	·031	460	·171	660	·509	860	1·119
162	·008	270	·035	470	·182	670	·533	870	1·158
169	·009	280	·039	480	·194	680	·556	880	1·198
175	·010	290	·043	490	·207	690	·580	890	1·240
182	·011	300	·048	500	·220	700	·604	900	1·282
187	·012	310	·052	510	·233	710	·632	910	1·326
192	·013	320	·057	520	·247	720	·660	920	1·370
197	·014	330	·063	530	·262	730	·687	930	1·415
202	·015	340	·069	540	·277	740	·715	940	1·461
206	·016	350	·075	550	·293	750	·743	950	1·507
210	·017	360	·082	560	·311	760	·774	960	1·555
215	·018	370	·090	570	·329	770	·805	970	1·602
219	·019	380	·097	580	·346	780	·837	980	1·652
223	·020	390	·105	590	·364	790	·869	990	1·704
227	·021	400	·113	600	·381	800	·901	1000	1·757

Note.—Enter first column with number equal to or next smaller than the given argument.

TABLE XVII.—*Concluded.*

Red. from tan. to arc. Arg. $= \Delta_0 a + \Delta_1 a + \Delta_2 a = \Delta_t a$.

$\Delta_t a$.	R.	$\Delta_t a$.	R.	$\Delta_t a$.	R.	$\Delta_t a$.	R.	$\Delta_t a$.	R.
s.	s.	s.	s.	s.	s.	s.	s.	s.	s.
1000	1·76	1250	3·43	1500	5·91	1750	9·36	2000	13·92
1005	1·79	1255	3·47	1505	5·96	1755	9·44	2005	14·03
1010	1·81	1260	3·51	1510	6·02	1760	9·52	2010	14·13
1015	1·84	1265	3·55	1515	6·08	1765	9·60	2015	14·24
1020	1·87	1270	3·59	1520	6·14	1770	9·68	2020	14·34
1025	1·89	1275	3·63	1525	6·20	1775	9·76	2025	14·45
1030	1·92	1280	3·68	1530	6·26	1780	9·84	2030	14·56
1035	1·95	1285	3·72	1535	6·32	1785	9·92	2035	14·66
1040	1·98	1290	3·76	1540	6·38	1790	10·01	2040	14·77
1045	2·01	1295	3·81	1545	6·44	1795	10·09	2045	14·88
1050	2·03	1300	3·85	1550	6·51	1800	10·18	2050	14·99
1055	2·06	1305	3·90	1555	6·57	1805	10·26	2055	15·10
1060	2·09	1310	3·94	1560	6·64	1810	10·35	2060	15·21
1065	2·12	1315	3·99	1565	6·70	1815	10·43	2065	15·32
1070	2·15	1320	4·03	1570	6·77	1820	10·52	2070	15·43
1075	2·18	1325	4·08	1575	6·83	1825	10·60	2075	15·54
1080	2·21	1330	4·12	1580	6·89	1830	10·69	2080	15·65
1085	2·24	1335	4·17	1585	6·96	1835	10·77	2085	15·76
1090	2·28	1340	4·22	1590	7·03	1840	10·86	2090	15·87
1095	2·31	1345	4·26	1595	7·10	1845	10·95	2095	15·99
1100	2·34	1350	4·31	1600	7·17	1850	11·04	2100	16·10
1105	2·37	1355	4·36	1605	7·23	1855	11·13	2105	16·22
1110	2·40	1360	4·41	1610	7·29	1860	11·22	2110	16·33
1115	2·43	1365	4·45	1615	7·36	1865	11·31	2115	16·45
1120	2·47	1370	4·50	1620	7·43	1870	11·40	2120	16·56
1125	2·50	1375	4·55	1625	7·50	1875	11·49	2125	16·68
1130	2·53	1380	4·60	1630	7·57	1880	11·58	2130	16·79
1135	2·56	1385	4·65	1635	7·64	1885	11·68	2135	16·91
1140	2·60	1390	4·70	1640	7·71	1890	11·77	2140	17·03
1145	2·63	1395	4·75	1645	7·78	1895	11·86	2145	17·15
1150	2·67	1400	4·81	1650	7·85	1900	11·96	2150	17·27
1155	2·70	1405	4·86	1655	7·92	1905	12·05	2155	17·39
1160	2·74	1410	4·91	1660	7·99	1910	12·14	2160	17·51
1165	2·77	1415	4·96	1665	8·06	1915	12·24	2165	17·63
1170	2·81	1420	5·01	1670	8·14	1920	12·33	2170	17·75
1175	2·85	1425	5·07	1675	8·21	1925	12·43	2175	17·87
1180	2·88	1430	5·12	1680	8·28	1930	12·53	2180	17·99
1185	2·92	1435	5·17	1685	8·35	1935	12·62	2185	18·12
1190	2·96	1440	5·23	1690	8·43	1940	12·72	2190	18·24
1195	2·99	1445	5·28	1695	8·50	1945	12·82	2195	18·36
1200	3·03	1450	5·34	1700	8·58	1950	12·92	2200	18·49
1205	3·07	1455	5·39	1705	8·65	1955	13·02	2205	18·61
1210	3·11	1460	5·45	1710	8·73	1960	13·11	2210	18·74
1215	3·15	1465	5·51	1715	8·81	1965	13·21	2215	18·86
1220	3·19	1470	5·56	1720	8·89	1970	13·31	2220	18·99
1225	3·22	1475	5·62	1725	8·96	1975	13·42	2225	19·12
1230	3·26	1480	5·67	1730	9·04	1980	13·52	2230	19·25
1235	3·30	1485	5·73	1735	9·12	1985	13·62	2235	19·38
1240	3·34	1490	5·79	1740	9·20	1990	13·72	2240	19·50
1245	3·39	1495	5·85	1745	9·28	1995	13·82	2245	19·63
1250	3·43	1500	5·91	1750	9·36	2000	13·92	2250	19·76

APPENDIX V.

Reduction of Struve-Peters Centennial Precessions to the Adopted Values.

$$\Delta p_\alpha = -0^{\cdot}035 \text{ s. } \sin\alpha\tan\delta - 0^{\cdot}038 \text{ s.}$$
$$\Delta p_\delta = -0''{\cdot}53 \cos\alpha.$$

TABLE XVIII.

Arg. R.A. h.	m.	−35 sin α.	−53 cos α.	m.	h.
0	0	− 0 −	−53 +	0	12
	10	− 2 −	−53	50	
	20	− 3 −	−53	40	
	30	− 5 −	−52	30	
	40	− 6 −	−52	20	
	50	− 8 −	−52	10	
1	0	− 9 −	−51 +	0	11
	10	−11 −	−50	50	
	20	−12 −	−50	40	
	30	−13 −	−49	30	
	40	−15 −	−48	20	
	50	−16 −	−46	10	
2	0	−18 −	−45 +	0	10
	10	−19 −	−44	50	
	20	−20 −	−43	40	
	30	−21 −	−42	30	
	40	−22 −	−41	20	
	50	−24 −	−39	10	
3	0	−25 −	−37 +	0	9
	10	−26 −	−36	50	
	20	−27 −	−34	40	
	30	−28 −	−32	30	
	40	−29 −	−30	20	
	50	−30 −	−28	10	
4	0	−30 −	−26 +	0	8
	10	−31 −	−24	50	
	20	−32 −	−22	40	
	30	−32 −	−20	30	
	40	−33 −	−18	20	
	50	−33 −	−16	10	
5	0	−34 −	−14 +	0	7
	10	−34 −	−11	50	
	20	−34 −	− 9	40	
	30	−35 −	− 7	30	
	40	−35 −	− 5	20	
	50	−35 −	− 2	10	
6	0	−35 −	− 0 +	0	6
				Arg.	

The algebraic sign is on the same side of the number as the argument.

When the R.A. exceeds 12 h. enter with Arg. R.A. − 12 h. and change the sign.

TABLE XIX.

δ.	Nat. tan δ.	δ.	Nat. tan δ.
°		°	
0	·000	45	1·000
1	·017	46	1·036
2	·035	47	1·072
3	·052	48	1·111
4	·070	49	1·150
5	·087	50	1·192
6	·105	51	1·235
7	·123	52	1·280
8	·141	53	1·327
9	·158	54	1·376
10	·176	55	1·428
11	·194	56	1·483
12	·213	57	1·540
13	·231	58	1·600
14	·249	59	1·664
15	·268	60	1·732
16	·287	61	1·804
17	·306	62	1·881
18	·325	63	1·963
19	·344	64	2·050
20	·364	65	2·145
21	·384	66	2·246
22	·404	67	2·356
23	·424	68	2·475
24	·445	69	2·605
25	·466	70	2·747
26	·488	71	2·904
27	·510	72	3·078
28	·532	73	3·271
29	·554	74	3·487
30	·577	75	3·732
31	·601	76	4·011
32	·625	77	4·331
33	·649	78	4·705
34	·675	79	5·145
35	·700	80	5·671
36	·727	81	6·314
37	·754	82	7·115
38	·781	83	8·144
39	·810	84	9·514
40	·839	85	11·430
41	·869	86	14·301
42	·900	87	19·081
43	·933	88	28·636
44	·966	89	57·290
45	1·000	90	∞

APPENDIX VI.

CONVERSION OF LONGITUDE AND LATITUDE INTO RIGHT ASCENSION AND DECLINATION, AND *VICE VERSA*.

$\epsilon = 23° \; 27' \; 0''$.

Precepts for the use of Table XX.:

To Convert Longitude and Latitude into Right Ascension and Declination:

$$\lambda \text{ or } \lambda - 180° = k,$$
$$\tan p = a \tan (\beta + B),$$
$$\tan \delta = b \tan (\beta + B) \cos p$$
$$\text{or} = b \sin (\beta + B),$$
$$\alpha = \lambda + A - p.$$

Rules for Algebraic Sign.

Sign of a is that of $\cos \lambda$,
,, B ,, $\sin \lambda$,
,, A ,, $\tan \lambda$,
,, b always $+$.

To Convert Right Ascension and Declination into Longitude and Latitude:

$$\alpha \text{ or } \alpha - 180° = k,$$
$$\tan q = a \tan (\delta - B),$$
$$\tan \beta = b \tan (\delta - B) \cos q$$
$$\text{or} = b \sin (\delta - B),$$
$$\lambda = \alpha + A + q.$$

Rules for Algebraic Sign.

Sign of a is that of $\cos \alpha$,
,, B ,, $\sin \alpha$,
,, A ,, $\tan \alpha$,
,, b always $+$.

The following approximate formula may be used when $\beta < 10°$:

$$\lambda = \alpha + A + a(\delta - B) \sec \beta,$$
$$\beta = b(\delta - B).$$

$\sec \beta$ may be put $= 1$ when $\beta < 4°$.

In using Table XXI. the algebraic sign of the coefficient is that on the same side as the argument.

TABLE XX.

Conversion of Longitude and Latitude into Right Ascension and Declination, and vice versa.

ε = 23° 27′ 0″.

k.	*k.*	*A.*	log *a.*	*a.*	*B.*	Diff.	log *b.*	*b.*	*k.*	*k.*
°	h. m.	° ′			° ′	′			h. m.	°
0	0 0	0 0·0	9·5998	0·3980	0 0·0	26·0	9·9626	0·9174	12 0	180
1	0 4	0 5·4	9·5998	0·3979	0 26·0	26·0	9·9626	0·9174	11 56	179
2	0 8	0 10·8	9·5996	0·3977	0 52·0	26·0	9·9626	0·9175	11 52	178
3	0 12	0 16·2	9·5992	0·3974	1 18·0	26·0	9·9627	0·9176	11 48	177
4	0 16	0 21·5	9·5988	0·3970	1 44·0	25·9	9·9628	0·9178	11 44	176
5	0 20	0 26·9	9·5982	0·3964	2 9·9	25·9	9·9629	0·9181	11 40	175
6	0 24	0 32·1	9·5974	0·3958	2 35·8	25·8	9·9630	0·9184	11 36	174
7	0 28	0 37·4	9·5966	0·3950	3 1·6	25·7	9·9632	0·9187	11 32	173
8	0 32	0 42·6	9·5956	0·3941	3 27·3	25·6	9·9634	0·9191	11 28	172
9	0 36	0 47·7	9·5944	0·3931	3 52·9	25·6	9·9636	0·9195	11 24	171
10	0 40	0 52·8	9·5932	0·3919	4 18·5	25·4	9·9638	0·9200	11 20	170
11	0 44	0 57·8	9·5918	0·3906	4 43·9	25·3	9·9640	0·9205	11 16	169
12	0 48	1 2·7	9·5902	0·3893	5 9·2	25·2	9·9643	0·9211	11 12	168
13	0 52	1 7·5	9·5885	0·3877	5 34·4	25·0	9·9646	0·9218	11 8	167
14	0 56	1 12·3	9·5867	0·3861	5 59·4	24·9	9·9649	0·9224	11 4	166
15	1 0	1 16·9	9·5848	0·3844	6 24·3	24·8	9·9653	0·9232	11 0	165
16	1 4	1 21·4	9·5827	0·3825	6 49·1	24·6	9·9656	0·9239	10 56	164
17	1 8	1 25·9	9·5804	0·3806	7 13·7	24·4	9·9660	0·9247	10 52	163
18	1 12	1 30·2	9·5780	0·3785	7 38·1	24·2	9·9664	0·9256	10 48	162
19	1 16	1 34·3	9·5755	0·3763	8 2·3	24·0	9·9668	0·9265	10 44	161
20	1 20	1 38·4	9·5728	0·3740	8 26·3	23·9	9·9673	0·9274	10 40	160
21	1 24	1 42·3	9·5700	0·3715	8 50·2	23·6	9·9677	0·9284	10 36	159
22	1 28	1 46·1	9·5670	0·3690	9 13·8	23·4	9·9682	0·9294	10 32	158
23	1 32	1 49·8	9·5639	0·3663	9 37·2	23·2	9·9687	0·9305	10 28	157
24	1 36	1 53·3	9·5606	0·3635	10 0·4	22·9	9·9692	0·9316	10 24	156
25	1 40	1 56·6	9·5571	0·3607	10 23·3	22·6	9·9697	0·9327	10 20	155
26	1 44	1 59·8	9·5535	0·3577	10 45·9	22·5	9·9703	0·9338	10 16	154
27	1 48	2 2·9	9·5497	0·3546	11 8·4	22·2	9·9708	0·9350	10 12	153
28	1 52	2 5·7	9·5458	0·3514	11 30·6	22·0	9·9714	0·9362	10 8	152
29	1 56	2 8·5	9·5416	0·3481	11 52·6	21·6	9·9720	0·9375	10 4	151
30	2 0	2 11·0	9·5374	0·3446	12 14·2	21·4	9·9725	0·9387	10 0	150
31	2 4	2 13·4	9·5329	0·3411	12 35·6	21·1	9·9731	0·9400	9 56	149
32	2 8	2 15·6	9·5282	0·3375	12 56·7	20·8	9·9737	0·9413	9 52	148
33	2 12	2 17·6	9·5234	0·3337	13 17·5	20·6	9·9744	0·9427	9 48	147
34	2 16	2 19·5	9·5184	0·3299	13 38·1	20·2	9 9750	0·9440	9 44	146
35	2 20	2 21·2	9·5132	0·3260	13 58·3	19·9	9·9756	0·9454	9 40	145
36	2 24	2 22·7	9·5078	0·3220	14 18·2	19·6	9·9762	0·9468	9 36	144
37	2 28	2 24·0	9·5022	0·3178	14 37·8	19·3	9·9769	0·9482	9 32	143
38	2 32	2 25·1	9·4964	0·3136	14 57·1	19·0	9·9775	0·9496	9 28	142
39	2 36	2 26·1	9·4903	0·3093	15 16·1	18·7	9·9782	0·9510	9 24	141
40	2 40	2 26·8	9·4841	0·3048	15 34·8	18·3	9·9788	0·9524	9 20	140
41	2 44	2 27·4	9·4776	0·3003	15 53·1	18·0	9·9795	0·9538	9 16	139
42	2 48	2 27·8	9·4709	0·2957	16 11·1	17·7	9·9801	0·9553	9 12	138
43	2 52	2 28·1	9·4640	0·2910	16 28·8	17·3	9·9808	0·9567	9 8	137
44	2 56	2 28·1	9·4568	0·2863	16 46·1	17·0	9·9814	0·9581	9 4	136
45	3 0	2 28·0	9·4493	0·2814	17 3·1		9·9821	0·9596	9 0	135

TABLE XX.—*Concluded.*

Conversion of Longitude and Latitude into Right Ascension and Declination, and vice versa.

$\epsilon = 23° 27' 0''$.

k.	k.	A.	log a.	a.	B.	Diff.	log b.	b.	k.	k.
°	h. m.	° ′			° ′	′			h. m.	°
45	3 0	2 28·0	9·4493	0·2814	17 3·1	16·7	9·9821	0·9596	9	135
46	3 4	2 27·7	9·4416	0·2764	17 19·8	16·3	9·9827	0·9610	8 56	134
47	3 8	2 27·2	9·4336	0·2714	17 36·1	15·9	9·9834	0·9625	8 52	133
48	3 12	2 26·5	9·4253	0·2663	17 52·0	15·6	9·9840	0·9639	8 48	132
49	3 16	2 25·7	9·4168	0·2611	18 7·6	15·3	9·9847	0·9653	8 44	131
50	3 20	2 24·7	9·4079	0·2558	18 22·9	14·9	9·9853	0·9667	8 40	130
51	3 24	2 23·5	9·3987	0·2504	18 37·8	14·5	9·9859	0·9681	8 36	129
52	3 28	2 22·1	9·3892	0·2450	18 52·3	14·2	9·9866	0·9695	8 32	128
53	8 32	2 20·6	9·3793	0·2395	19 6·5	13·8	9·9872	0·9709	8 28	127
54	3 36	2 18·9	9·3690	0·2339	19 20·3	13·4	9·9878	0·9723	8 24	126
55	3 40	2 17·1	9·3584	0·2283	19 33·7	13·1	9·9884	0·9736	8 20	125
56	3 44	2 15·1	9·3474	0·2225	19 46·8	12·7	9·9890	0·9749	8 16	124
57	3 48	2 12·9	9·3359	0·2167	19 59·5	12·3	9·9896	0·9762	8 12	123
58	3 52	2 10·6	9·3240	0·2109	20 11·8	12·0	9·9901	0·9775	8 8	122
59	3 56	2 8·1	9·3117	0·2050	20 23·8	11·6	9·9907	0·9788	8 4	121
60	4 0	2 5·5	9·2988	0·1990	20 35·4	11·2	9·9912	0·9800	8 0	120
61	4 4	2 2·7	9·2854	0·1929	20 46·6	10·8	9·9918	0·9812	7 56	119
62	4 8	1 59·8	9·2714	0·1868	20 57·4	10·5	9·9923	0·9824	7 52	118
63	4 12	1 56·8	9·2569	0·1807	21 7·9	10·1	9·9928	0·9835	7 48	117
64	4 16	1 53·6	9·2417	0·1744	21 18·0	9·7	9·9933	0·9847	7 44	116
65	4 20	1 50·3	9·2258	0·1682	21 27·7	9·3	9·9938	0·9858	7 40	115
66	4 24	1 46·9	9·2091	0·1619	21 37·0	9·0	9·9942	0·9868	7 36	114
67	4 28	1 43·4	9·1917	0·1555	21 46·0	8·6	9·9947	0·9878	7 32	113
68	4 32	1 39·7	9·1734	0·1491	21 54·6	8·2	9·9951	0·9888	7 28	112
69	4 36	1 36·0	9·1542	0·1426	22 2·8	7·8	9·9955	0·9898	7 24	111
70	4 40	1 32·1	9·1339	0·1361	22 10·6	7·4	9·9959	0·9907	7 20	110
71	4 44	1 28·2	9·1125	0·1296	22 18·0	7·1	9·9963	0·9916	7 16	109
72	4 48	1 24·1	9·0898	0·1230	22 25·1	6·7	9·9967	0·9924	7 12	108
73	4 52	1 19·9	9·0658	0·1164	22 31·8	6·3	9·9970	0·9932	7 8	107
74	4 56	1 15·7	9·0402	0·1097	22 38·1	5·9	9·9974	0·9940	7 4	106
75	5 0	1 11·4	9·0128	0·1030	22 44·0	5·5	9·9977	0·9947	7 0	105
76	5 4	1 7·0	8·9835	0·0963	22 49·5	5·2	9·9980	0·9954	6 56	104
77	5 8	1 2·5	8·9519	0·0895	22 54·7	4·8	9·9983	0·9960	6 52	103
78	5 12	0 57·9	8·9177	0·0827	22 59·5	4·4	9·9985	0·9966	6 48	102
79	5 16	0 53·3	8·8804	0·0759	23 3·9	4·0	9·9987	0·9971	6 44	101
80	5 20	0 48·7	8·8395	0·0691	23 7·9	3·6	9·9990	0·9976	6 40	100
81	5 24	0 43·9	8·7942	0·0623	23 11·5	3·3	9·9992	0·9981	6 36	99
82	5 28	0 39·2	8·7434	0·0554	23 14·8	2·8	9·9993	0·9985	6 32	98
83	5 32	0 34·4	8·6857	0·0485	23 17·6	2·5	9·9995	0·9988	6 28	97
84	5 36	0 29·5	8·6191	0·0416	23 20·1	2·1	9·9996	0·9991	6 24	96
85	5 40	0 24·7	8·5401	0·0347	23 22·2	1·7	9·9997	0·9994	6 20	95
86	5 44	0 19·8	8·4434	0·0278	23 23·9	1·4	9·9998	0·9996	6 16	94
87	5 48	0 14·8	8·3186	0·0208	23 25·3	0·9	9·9999	0·9998	6 12	93
88	5 52	0 9·9	8·1426	0·0139	23 26·2	0·6	0·0000	0·9999	6 8	92
89	5 56	0 5·0	7·8417	0·0069	23 26·8	0·2	0·0000	1·0000	6 4	91
90	6 0	0 0·0	$-\infty$	0·0000	23 27·0		0·0000	1·0000	6 0	90

TABLE XXI.

Factors for converting Small Changes of Latitude and Longitude near the Ecliptic into Changes of Right Ascension and Declination.

Formulae : $\delta\alpha = \delta v + (v\alpha)\,\delta v + (\beta\alpha)\,\delta\beta,$
$\delta\delta = \delta\beta + (v\delta)\,\delta v + (\beta\delta)\,\delta\beta.$

Longitude.	$(v\alpha)$.			$(\beta\alpha)$.			$(v\delta)$.	$(\beta\delta)$.	
	$\beta=-5°$.	0°.	+5°.	$\beta=-5°$.	0°.	+5°.			
°									°
270	+ ·133 +	+ ·090 +	+ ·050 +	·000	·000	·000	·000	·000	270
275	·131	·089	·049	− ·043 +	− ·041 +	− ·040 +	+ ·038 −	− ·001 −	265
280	·126 +	·084 +	·045 +	− ·085 +	− ·081 +	− ·079 +	·075 −	− ·003 −	260
285	+ ·117	+ ·076	+ ·039	− ·126	− ·121	− ·117	+ ·111 −	− ·006 −	255
290	·105	·066	·030	− ·165	− ·158	− ·154	·147 −	− ·011 −	250
295	·091 +	·055 +	·021 +	− ·202 +	− ·193 +	− ·188 +	·180 −	− ·016 −	245
300	+ ·074	+ ·041	+ ·009 +	− ·235	− ·226	− ·220	+ ·212 −	− ·023 −	240
305	·057	·027	− ·003 −	− ·265	− ·256	− ·250	·242 −	− ·030 −	235
310	·039 +	+ ·011 +	− ·016 −	− ·292 +	− ·282 +	− ·277 +	·269 −	− ·037 −	230
315	+ ·021	− ·004 −	− ·028 −	− ·316	− ·306	− ·301	+ ·293 −	− ·044 −	225
320	+ ·003 +	− ·018 −	− ·040 −	− ·336	− ·326	− ·322	·315 −	− ·051 −	220
325	− ·014 −	− ·032 −	− ·052 −	− ·353 +	− ·344 +	− ·340 +	·335 −	− ·058 −	215
330	− ·030 −	− ·045 −	− ·061 −	− ·368	− ·359	− ·356	+ ·352 −	− ·064 −	210
335	− ·044 −	− ·056 −	− ·070 −	− ·379	− ·371	− ·369	·366 −	− ·069 −	205
340	− ·056 −	− ·065 −	− ·077 −	− ·388 +	− ·381 +	− ·379 +	·378 −	− ·074 −	200
345	− ·066 −	− ·073 −	− ·082 −	− ·395	− ·388	− ·388	+ ·386 −	− ·078 −	195
350	− ·074 −	− ·078 −	− ·085 −	− ·399	− ·394	− ·394	·393 −	− ·080 −	190
355	− ·080 −	− ·081 −	− ·085 −	− ·401 +	− ·397 +	− ·399 +	·397 −	− ·082 −	185
0	− ·084 −	− ·083 −	− ·084 −	− ·401	− ·398	− ·401	+ ·398 −	− ·082 −	180
5	− ·085 −	− ·081 −	− ·080 −	− ·399	− ·397	− ·401	·397 −	− ·082 −	175
10	− ·085 −	− ·078 −	− ·074 −	− ·394 +	− ·394 +	− ·399 +	·393 −	− ·080 −	170
15	− ·082 −	− ·073 −	− ·066 −	− ·388	− ·388	− ·395	+ ·386 −	− ·078 −	165
20	− ·077 −	− ·065 −	− ·056 −	− ·379	− ·381	− ·388	·378 −	− ·074 −	160
25	− ·070 −	− ·056 −	− ·044 −	− ·369 +	− ·371 +	− ·379 +	·366 −	− ·069 −	155
30	− ·061 −	− ·045 −	− ·030 −	− ·356	− ·359	− ·368	+ ·352 −	− ·064 −	150
35	− ·052 −	− ·032 −	− ·014 −	− ·340	− ·344	− ·353	·335 −	− ·058 −	145
40	− ·040 −	− ·018 −	+ ·003 +	− ·322 +	− ·326 +	− ·336 +	·315 −	− ·051 −	140
45	− ·028 −	− ·004 −	+ ·021	− ·301	− ·306	− ·316	+ ·293 −	− ·044 −	135
50	− ·016 −	+ ·011 +	·039	− ·277	− ·282	− ·292	·269 −	− ·037 −	130
55	− ·003 −	+ ·027 +	·057 +	− ·250 +	− ·256 +	− ·265 +	·242 −	− ·030 −	125
60	+ ·009 +	+ ·041	+ ·074	− ·220	− ·226	− ·235	+ ·212 −	− ·023 −	120
65	·021	·055	·091	− ·188	− ·193	− ·202	·180 −	− ·016 −	115
70	·030 +	·066 +	·105 +	− ·154 +	− ·158 +	− ·165 +	·147 −	− ·011 −	110
75	+ ·039	+ ·076	+ ·117	− ·117	− ·121	− ·126	+ ·111 −	− ·006 −	105
80	·045	·084	·126	− ·079	− ·081	− ·085	·075 −	− ·003 −	100
85	·049 +	·089 +	·131 +	− ·040 +	− ·041 +	− ·043 +	·038 −	− ·001 −	95
90	+ ·050 +	+ ·090 +	+ ·133 +	·000	·000	·000	·000	·000	90
	$\beta=-5°$.	0°.	+5°.	$\beta=-5°$.	0°.	+5°.			Longitude.

APPENDIX VII.

TABLE XXII.

Table of Refractions for 50° *F. Temp. and* 30 *in. Pressure.*

App. Z.D.	Mean Refraction.	Change for 10° F. Temp.	Change for 1 in. Bar.	App. Z.D.	Mean Refraction.	Change for 10° F. Temp.	Change for 1 in. Bar.
°	"	"	"	°	' "	"	"
0	0·00	0·00	0·00	45	0 58·2	− 1·1	+ 2·0
1	1·02	−0·02	+0·03	46	1 0·2	− 1·2	+ 2·1
2	2·03	−0·04	+0·07	47	1 2·4	− 1·3	+ 2·1
3	3·05	−0·06	+0·10	48	1 4·6	− 1·3	+ 2·2
4	4·07	−0·08	+0·14	49	1 6·9	− 1·3	+ 2·2
5	5·09	−0·10	+0·17	50	1 9·3	− 1·4	+ 2·3
6	6·12	−0·12	+0·21	51	1 11·8	− 1·4	+ 2·4
7	7·15	−0·14	+0·24	52	1 14·4	− 1·5	+ 2·5
8	8·18	−0·16	+0·28	53	1 17·1	− 1·5	+ 2·6
9	9·22	−0·18	+0·31	54	1 20·0	− 1·6	+ 2·7
10	10·27	−0·21	+0·34	55	1 23·0	− 1·6	+ 2·8
11	11·32	−0·23	+0·38	56	1 26·1	− 1·7	+ 2·9
12	12·38	−0·25	+0·42	57	1 29·4	− 1·7	+ 3·0
13	13·44	−0·26	+0·45	58	1 32·9	− 1·8	+ 3·2
14	14·52	−0·29	+0·49	59	1 36·6	− 1·9	+ 3·3
15	15·60	−0·31	+0·53	60	1 40·5	− 2·0	+ 3·4
16	16·70	−0·33	+0·56	61	1 44·6	− 2·1	+ 3·6
17	17·80	−0·35	+0·60	62	1 49·1	− 2·2	+ 3·7
18	18·92	−0·37	+0·64	63	1 53·8	− 2·2	+ 3·9
19	20·04	−0·39	+0·68	64	1 58·8	− 2·3	+ 4·0
20	21·19	−0·42	+0·72	65	2 4·2	− 2·4	+ 4·2
21	22·35	−0·44	+0·76	66	2 10·0	− 2·5	+ 4·4
22	23·52	−0·46	+0·80	67	2 16·3	− 2·6	+ 4·6
23	24·71	−0·48	+0·84	68	2 23·1	− 2·8	+ 4·9
24	25·92	−0·51	+0·88	69	2 30·5	− 2·9	+ 5·1
25	27·15	−0·54	+0·92	70	2 38·6	− 3·1	+ 5·4
26	28·39	−0·56	+0·97	71	2 47·5	− 3·3	+ 5·7
27	29·66	−0·58	+1·01	72	2 57·3	− 3·5	+ 6·0
28	30·95	−0·60	+1·05	73	3 8·2	− 3·7	+ 6·4
29	32·26	−0·63	+1·10	74	3 20·3	− 4·0	+ 6·8
30	33·60	−0·65	+1·15	75	3 33·9	− 4·3	+ 7·3
31	34·97	−0·69	+1·19	76	3 49·4	− 4·6	+ 7·8
32	36·37	−0·72	+1·23	77	4 7·0	− 5·0	+ 8·3
33	37·79	−0·74	+1·28	78	4 27·4	− 5·5	+ 9·0
34	39·26	−0·77	+1·33	79	4 51·1	− 6·0	+ 9·9
35	40·75	−0·80	+1·38	80	5 19·0	− 6·5	+10·9
36	42·28	−0·83	+1·43	81	5 52·5	− 7·3	+12·0
37	43·84	−0·86	+1·48	82	6 33·2	− 8·2	+13·4
38	45·46	−0·89	+1·54	83	7 23·7	− 9·3	+15·3
39	47·12	−0·92	+1·60	84	8 27·7	−10·8	+17·5
40	48·82	−0·96	+1·66	85	9 51·4	−12·9	+20·4
41	50·57	−0·99	+1·72	86	11 44·3	−16·0	+24·3
42	52·37	−1·02	+1·78	87	14 22·6	−20·6	+30·0
43	54·24	−1·07	+1·84	88	18 16·1	−28·3	+38·6
44	56·17	−1·10	+1·91	89	24 20·6	−41·7	+52·4
45	58·16	−1·14	+1·97	90	34 32·1	−68·6	+76·5

APPENDIX VIII.

COEFFICIENTS FOR THE NUTATION AND THE RELATED STAR CONSTANTS.

Ω, Longitude of Moon's node.
ω, Distance from node to lunar perigee.
g, Moon's mean anomaly.
☾, Moon's mean longitude $= g + \omega + \Omega$.
D, Mean elongation of Moon from Sun.
ω', Distance from Moon's node to solar perigee.
g', Sun's mean anomaly.
L, Sun's mean longitude.
T, Time after 1900 in centuries.

Each number in the column $\delta\psi$ is the coefficient of the sine of the corresponding argument in the expression of $\delta\psi$, and each in column $\delta\epsilon$ is the coefficient of the cosine of the argument in the expression for the obliquity. The two remaining columns give the corresponding coefficients for the star constants, A and B.

Action of the Moon.

Argument.	$\delta\psi$.	$\delta\epsilon$.	A.	B.
	sin ″	cos ″	sin ″	cos ″
Ω	− 17·234 − 0·017 T	+ 9·210 + 0·0009 T	− 0·34215 − 0·00030 T	− 9·210 − 0·0009 T
2Ω	+ ·209	− ·090	+ ·00415	+ ·090
2☾	− ·204	+ ·089	− ·00405	− ·089
g	+ ·068	·000	+ ·00135	·000
2☾ $- \Omega$	− ·034	+ ·018	− ·00068	− ·018
2☾ $+ g$	− ·026	+ ·011	− ·00052	− ·011
$2D - g$	+ ·015	·000	+ ·00030	·000
$2L - \Omega$	+ ·012	− ·007	+ ·00025	+ ·007
2☾ $- g$	+ ·011	− ·005	+ ·00023	+ ·005
$2D$	+ ·006	·000	+ ·00012	·000
$g + \Omega$	+ ·006	− ·003	+ ·00012	+ ·003
$-g + \Omega$	− ·006	+ ·003	− ·00011	− ·003
$2\omega + \Omega$	+ ·005	− ·003	+ ·00010	+ ·003
$2D + 2$☾ $- g$	− ·005	+ ·002	− ·00010	− ·002
$2D - 2g$	− ·004	·000	− ·00009	·000
2☾ $+ g - \Omega$	− ·004	+ ·002	− ·00009	− ·002
$2D + 2$☾	− ·003	·000	− ·00006	·000
$2g$	+ ·003	·000	+ ·00006	·000
2☾ $+ 2g$	− ·003	·000	− ·00005	·000
$2L + g$	+ ·003	·000	+ ·00005	·000
$2L$	− ·002	·000	− ·00004	·000

Action of the Sun.

Argument.	$\delta\psi$.	$\delta\epsilon$.	A.	B.
	sin ″	cos ″	sin ″	cos ″
$2L$	− 1·270	+ 0·551	− 0·02521	− 0·551
g'	+ ·128	·000	+ ·00254	·000
$2L + g'$	− ·050	+ ·022	− ·00099	− ·022
$2L - g'$	+ ·021	− ·009	+ 0·0042	+ ·009

APPENDIX IX.

TABLE XXIII.

Three-Place Logarithms.

1	·000	301	51	·708	8	101	·004	5	151	·179	3
2	·301	176	52	·716	8	102	·009	4	152	·182	3
3	·477	125	53	·724	8	103	·013	4	153	·185	3
4	·602	97	54	·732	8	104	·017	4	154	·188	2
5	·699	79	55	·740	8	105	·021	4	155	·190	3
6	·778	67	56	·748	8	106	·025	4	156	·193	3
7	·845	58	57	·756	7	107	·029	4	157	·196	3
8	·903	51	58	·763	8	108	·033	4	158	·199	2
9	·954	46	59	·771	7	109	·037	4	159	·201	3
10	·000	41	60	·778	7	110	·041	4	160	·204	3
11	·041	38	61	·785	7	111	·045	4	161	·207	3
12	·079	35	62	·792	7	112	·049	4	162	·210	2
13	·114	32	63	·799	7	113	·053	4	163	·212	3
14	·146	30	64	·806	7	114	·057	4	164	·215	2
15	·176	28	65	·813	7	115	·061	3	165	·217	3
16	·204	26	66	·820	6	116	·064	4	166	·220	3
17	·230	25	67	·826	7	117	·068	4	167	·223	2
18	·255	24	68	·833	6	118	·072	4	168	·225	3
19	·279	22	69	·839	6	119	·076	3	169	·228	2
20	·301	21	70	·845	6	120	·079	4	170	·230	3
21	·322	20	71	·851	6	121	·083	3	171	·233	3
22	·342	20	72	·857	6	122	·086	4	172	·236	2
23	·362	18	73	·863	6	123	·090	3	173	·238	3
24	·380	18	74	·869	6	124	·093	4	174	·241	2
25	·398	17	75	·875	6	125	·097	3	175	·243	3
26	·415	16	76	·881	5	126	·100	4	176	·246	2
27	·431	16	77	·886	6	127	·104	3	177	·248	2
28	·447	15	78	·892	6	128	·107	4	178	·250	3
29	·462	15	79	·898	5	129	·111	3	179	·253	2
30	·477	14	80	·903	5	130	·114	3	180	·255	3
31	·491	14	81	·908	6	131	·117	4	181	·258	2
32	·505	14	82	·914	5	132	·121	3	182	·260	2
33	·519	12	83	·919	5	133	·124	3	183	·262	3
34	·531	13	84	·924	5	134	·127	3	184	·265	2
35	·544	12	85	·929	5	135	·130	4	185	·267	3
36	·556	12	86	·934	6	136	·134	3	186	·270	2
37	·568	12	87	·940	4	137	·137	3	187	·272	2
38	·580	11	88	·944	5	138	·140	3	188	·274	2
39	·591	11	89	·949	5	139	·143	3	189	·276	3
40	·602	11	90	·954	5	140	·146	3	190	·279	2
41	·613	10	91	·959	5	141	·149	3	191	·281	2
42	·623	10	92	·964	4	142	·152	3	192	·283	3
43	·633	10	93	·968	5	143	·155	3	193	·286	2
44	·643	10	94	·973	5	144	·158	3	194	·288	2
45	·653	10	95	·978	4	145	·161	3	195	·290	2
46	·663	9	96	·982	5	146	·164	3	196	·292	2
47	·672	9	97	·987	4	147	·167	3	197	·294	3
48	·681	9	98	·991	5	148	·170	3	198	·297	2
49	·690	9	99	·996	4	149	·173	3	199	·299	2
50	·699		100	·000		150	·176		200	·301	

TABLE XXIV.

Logarithmic Sines, etc., for every Degree.

°	Sin.	Diff.	Tan.	Diff.	Cot.	Cos.	°
0							90
1	8·242	301	8·242	301	1·758	0·000	89
2	8·543	176	8·543	176	1·457	0·000	88
3	8·719	125	8·719	126	1·281	9·999	87
4	8·844	96	8·845	97	1·155	9·999	86
5	8·940	79	8·942	80	1·058	9·998	85
6	9·019	67	9·022	67	0·978	9·998	84
7	9·086	58	9·089	59	0·911	9·997	83
8	9·144	50	9·148	52	0·852	9·996	82
9	9·194	46	9·200	46	0·800	9·995	81
10	9·240	41	9·246	43	0·754	9·993	80
11	9·281	37	9·289	38	0·711	9·992	79
12	9·318	34	9·327	36	0·673	9·990	78
13	9·352	32	9·363	34	0·637	9·989	77
14	9·384	29	9·397	31	0·603	9·987	76
15	9·413	27	9·428	29	0·572	9·985	75
16	9·440	26	9·457	28	0·543	9·983	74
17	9·466	24	9·485	27	0·515	9·981	73
18	9·490	23	9·512	25	0·488	9·978	72
19	9·513	21	9·537	24	0·463	9·976	71
20	9·534	20	9·561	23	0·439	9·973	70
21	9·554	20	9·584	22	0·416	9·970	69
22	9·574	18	9·606	22	0·394	9·967	68
23	9·592	17	9·628	21	0·372	9·964	67
24	9·609	17	9·649	20	0·351	9·961	66
25	9·626	16	9·669	19	0·331	9·957	65
26	9·642	15	9·688	19	0·312	9·954	64
27	9·657	15	9·707	19	0·293	9·950	63
28	9·672	14	9·726	18	0·274	9·946	62
29	9·686	13	9·744	17	0·256	9·942	61
30	9·699	13	7·761	18	0·239	9·938	60
31	9·712	12	9·779	17	0·221	9·933	59
32	9·724	12	9·796	17	0·204	9·928	58
33	9·736	12	9·813	16	0·187	9·924	57
34	9·748	11	9·829	16	0·171	9·919	56
35	9·759	10	9·845	16	0·155	9·913	55
36	9·769	10	9·861	16	0·139	9·908	54
37	9·779	10	9·877	16	0·123	9·902	53
38	9·789	10	9·893	15	0·107	9·897	52
39	9·799	9	9·908	16	0·092	9·891	51
40	9·808	9	9·924	15	0·076	9·884	50
41	9·817	9	9·939	15	0·061	9·878	49
42	9·826	8	9·954	16	0·046	9·871	48
43	9·834	8	9·970	15	0·030	9·864	47
44	9·842	7	9·985	15	0·015	9·857	46
45	9·849		0·000		0·000	9·849	45
°	Cos.		Cot.		Tan.	Sin.	°

TABLE XXV.

Logarithmic Sines, etc., for every Tenth of a Degree to 5°.

°	Sin.	Diff.	Tan.	Diff.	Cot.	Cos.	°
0·0						0·000	90·0
0·1	7·242	301	7·242	301	2·758	0·000	89·9
0·2	7·543	176	7·543	176	2·457	0·000	89·8
0·3	7·719	125	7·719	125	2·281	0·000	89·7
0·4	7·844	97	7·844	97	2·156	0·000	89·6
0·5	7·941	79	7·941	79	2·059	0·000	89·5
0·6	8·020	67	8·020	67	1·980	0·000	89·4
0·7	8·087	58	8·087	58	1·913	0·000	89·3
0·8	8·145	51	8·145	51	1·855	0·000	89·2
0·9	8·196	46	8·196	46	1·804	0·000	89·1
1·0	8·242	41	8·242	41	1·758	0·000	89·0
1·1	8·283	38	8·283	38	1·717	0·000	88·9
1·2	8·321	35	8·321	35	1·679	0·000	88·8
1·3	8·356	32	8·356	32	1·644	0·000	88·7
1·4	8·388	30	8·388	30	1·612	0·000	88·6
1·5	8·418	28	8·418	28	1·582	0·000	88·5
1·6	8·446	26	8·446	26	1·554	0·000	88·4
1·7	8·472	25	8·472	25	1·528	0·000	88·3
1·8	8·497	24	8·497	24	1·503	0·000	88·2
1·9	8·521	22	8·521	22	1·479	0·000	88·1
2·0	8·543	21	8·543	21	1·457	0·000	88·0
2·1	8·564	20	8·564	21	1·436	0·000	87·9
2·2	8·584	19	8·585	19	1·415	0·000	87·8
2·3	8·603	19	8·604	18	1·396	0·000	87·7
2·4	8·622	18	8·622	18	1·378	0·000	87·6
2·5	8·640	17	8·640	17	1·360	0·000	87·5
2·6	8·657	16	8·657	17	1·343	0·000	87·4
2·7	8·673	16	8·674	15	1·326	0·000	87·3
2·8	8·689	15	8·689	16	1·311	9·999	87·2
2·9	8·704	15	8·705	14	1·295	9·999	87·1
3·0	8·719	14	8·719	15	1·281	9·999	87·0
3·1	8·733	14	8·734	13	1·266	9·999	86·9
3·2	8·747	13	8·747	14	1·253	9·999	86·8
3·3	8·760	13	8·761	13	1·239	9·999	86·7
3·4	8·773	13	8·774	12	1·226	9·999	86·6
3·5	8·786	12	8·786	13	1·214	9·999	86·5
3·6	8·798	12	8·799	12	1·201	9·999	86·4
3·7	8·810	11	8·811	11	1·189	9·999	86·3
3·8	8·821	12	8·822	12	1·178	9·999	86·2
3·9	8·833	11	8·834	11	1·166	9·999	86·1
4·0	8·844	10	8·845	10	1·155	9·999	86·0
4·1	8·854	11	8·855	11	1·145	9·999	85·9
4·2	8·865	10	8·866	10	1·134	9·999	85·8
4·3	8·875	10	8·876	10	1·124	9·999	85·7
4·4	8·885	10	8·886	10	1·114	9·999	85·6
4·5	8·895	9	8·896	10	1·104	9·999	85·5
4·6	8·904	9	8·906	9	1·094	9·999	85·4
4·7	8·913	10	8·915	9	1·085	9·999	85·3
4·8	8·923	9	8·924	9	1·076	9·998	85·2
4·9	8·932	8	8·933	9	1·067	9·998	85·1
5·0	8·940		8·942		1·058	9·998	85·0
°	Cos.	Diff.	Cot.	Diff.	Tan.	Sin.	°

TABLE XXVI.

Natural Values of the Trigonometrical Functions.

°	Sin.	Diff.	Tan.	Diff.	Sec.	Diff.	°
0	·000	17	·000	17	1·000	0	90
1	·017	18	·017	18	1·000	1	89
2	·035	17	·035	17	1·001	0	88
3	·052	18	·052	18	1·001	1	87
4	·070	17	·070	17	1·002	2	86
5	·087	18	·087	18	1·004	2	85
6	·105	17	·105	18	1·006	2	84
7	·122	17	·123	18	1·008	2	83
8	·139	17	·141	17	1·010	2	82
9	·156	18	·158	18	1·012	3	81
10	·174	17	·176	18	1·015	4	80
11	·191	17	·194	19	1·019	3	79
12	·208	17	·213	18	1·022	4	78
13	·225	17	·231	18	1·026	5	77
14	·242	17	·249	19	1·031	4	76
15	·259	17	·268	19	1·035	5	75
16	·276	16	·287	19	1·040	6	74
17	·292	17	·306	19	1·046	5	73
18	·309	17	·325	19	1·051	7	72
19	·326	16	·344	20	1·058	6	71
20	·342	16	·364	20	1·064	7	70
21	·358	17	·384	20	1·071	8	69
22	·375	16	·404	20	1·079	7	68
23	·391	16	·424	21	1·086	9	67
24	·407	16	·445	21	1·095	8	66
25	·423	15	·466	22	1·103	10	65
26	·438	16	·488	22	1·113	9	64
27	·454	15	·510	22	1·122	11	63
28	·469	16	·532	22	1·133	10	62
29	·485	15	·554	23	1·143	12	61
30	·500	15	·577	24	1·155	12	60
31	·515	15	·601	24	1·167	12	59
32	·530	15	·625	24	1·179	13	58
33	·545	14	·649	26	1·192	14	57
34	·559	15	·675	25	1·206	15	56
35	·574	14	·700	27	1·221	15	55
36	·588	14	·727	27	1·236	16	54
37	·602	14	·754	27	1·252	17	53
38	·616	13	·781	29	1·269	18	52
39	·629	14	·810	29	1·287	18	51
40	·643	13	·839	30	1·305	20	50
41	·656	13	·869	31	1·325	21	49
42	·669	13	·900	33	1·346	21	48
43	·682	13	·933	33	1·367	23	47
44	·695	12	·966	34	1·390	24	46
45	·707		1·000		1·414		45
	Cos.	Diff.	Cot.	Diff.	Cosec.	Diff.	

TABLE XXVI.—*Concluded.*

Natural Values of the Trigonometrical Functions.

°	Cosec.	Diff.	Cot.	Diff.	Cos.	Diff.	°
0					1·000	0	90
1	57·299	28·6	57·290	28·6	1·000	1	89
2	28·654	9·94	28·636	9·55	·999	0	88
3	19·107	4·77	19·081	4·78	·999	1	87
4	14·336	2·86	14·301	2·87	·998	2	86
5	11·474	1·91	11·430	1·92	·996	1	85
6	9·567	1·36	9·514	1·37	·995	2	84
7	8·206	1·02	8·144	1·03	·993	3	83
8	7·185	·793	7·115	·801	·990	2	82
9	6·392	·633	6·314	·643	·988	3	81
10	5·759	·518	5·671	·526	·985	3	80
11	5·241	·431	5·145	·440	·982	4	79
12	4·810	·365	4·705	·374	·978	4	78
13	4·445	·311	4·331	·320	·974	4	77
14	4·134	·270	4·011	·279	·970	4	76
15	3·864	·236	3·732	·245	·966	5	75
16	3·628	·208	3·487	·216	·961	5	74
17	3·420	·184	3·271	·193	·956	5	73
18	3·236	·164	3·078	·174	·951	5	72
19	3·072	·148	2·904	·157	·946	6	71
20	2·924	·134	2·747	·142	·940	6	70
21	2·790	·121	2·605	·130	·934	7	69
22	2·669	·110	2·475	·119	·927	6	68
23	2·559	·100	2·356	·110	·921	7	67
24	2·459	·093	2·246	·101	·914	8	66
25	2·366	·085	2·145	·095	·906	7	65
26	2·281	·078	2·050	·087	·899	8	64
27	2·203	·073	1·963	·082	·891	8	63
28	2·130	·067	1·881	·077	·883	8	62
29	2·063	·063	1·804	·072	·875	9	61
30	2·000	·058	1·732	·068	·866	9	60
31	1·942	·055	1·664	·064	·857	9	59
32	1·887	·051	1·600	·060	·848	9	58
33	1·836	·048	1·540	·057	·839	10	57
34	1·788	·045	1·483	·055	·829	10	56
35	1·743	·042	1·428	·052	·819	10	55
36	1·701	·039	1·376	·049	·809	10	54
37	1·662	·038	1·327	·047	·799	11	53
38	1·624	·035	1·280	·045	·788	11	52
39	1·589	·033	1·235	·043	·777	11	51
40	1·556	·032	1·192	·042	·766	11	50
41	1·524	·030	1·150	·039	·755	12	49
42	1·494	·028	1·111	·039	·743	12	48
43	1·466	·026	1·072	·036	·731	12	47
44	1·440	·026	1·036	·036	·719	12	46
45	1·414		1·000		·707		45
	Sec.	Diff.	Tan.	Diff.	Sin.	Diff.	

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